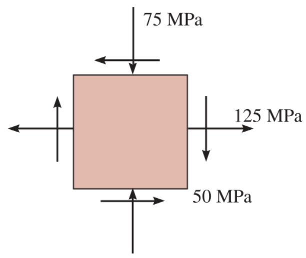
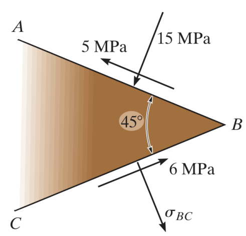
Tutorial Sheet 4:
Transformation of Stress

1. Determine the equivalent state of stress on an element at the same point which represents (a) the principal stress, and (b) the maximum in-plane shear stress and the associated average normal stress. Also, for each case, determine the corresponding orientation of the element with respect to the element shown.

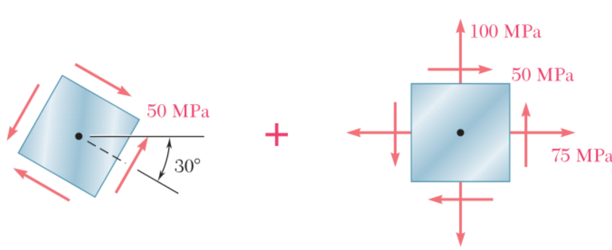
[$\sigma_{p1} = 137$ MPa, $\theta_{p1} = -13.3°$,
$\sigma_{p2} = -86.8$ MPa, $\theta_{p2} = 76.7°$,
$\tau_{\max,\text{in-plane}} = 112$ MPa, $\theta_s = -31.7°$ and $122°$,
$\sigma_{\text{avg}} = 25$ MPa]



2. Planes AB and BC at a point are subjected to the stresses shown in the figure. Determine the principal stresses acting at this point and find σ_{BC}. [$\sigma_{p1} = -1.19$ MPa, $\sigma_{p2} = -16.8$ MPa, $\sigma_{BC} = -14.1$ MPa]



3. Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown. [168.6 MPa, 6.42 MPa, 33.8°, 123.8°]



4. Consider the general 3D state of stress. The problem of finding the principal stresses reduces to the eigenvalue problem (in matrix form):

$$\begin{bmatrix} \sigma_{xx}-\sigma_p & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma_p & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz}-\sigma_p \end{bmatrix} \begin{bmatrix} n_x^p \\ n_y^p \\ n_z^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where σ_p denotes the principal stress and n_x^p, n_y^p, and n_z^p denote the components of the unit normal to the plane on which the principal stress occurs.

   (a) Show that for non-trivial solutions, an equation of the following form is satisfied: $\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$. What are the expressions of I_1, I_2, and I_3?

   (b) I_1, I_2, and I_3 are referred to as the stress invariants. What do you think is the motivation behind calling them "invariants", i.e. unchanging quantities?

5. (*Question for thinking!*) Go back to your notes/textbook of 1st year Linear Algebra from MA11004, and revise the following two theorems:

   (a) *Eigenvalues of a real, symmetric matrix must be real.*

   (b) *Eigenvectors corresponding to distinct eigenvalues must be perpendicular to each other.*

   What are the implications of these two theorems for the general 3D state of stress, the principal stresses, and the principal directions?

6. If we choose to orient our coordinate axes along the principal directions (see the previous problem), and the values of the principal stresses are σ_{p1}, σ_{p2}, and σ_{p3}, then write down the stress matrix. *(Please think long and hard about this question.)*

7. Consider a situation where the coordinate axes are oriented along the principal directions. Next consider a plane which is equally inclined to the coordinate axes. Let the unit outward normal to this plane is $\hat{\mathbf{n}} = [n_1\ n_2\ n_3]^\mathsf{T}$.

   (a) Show that $|n_1| = |n_2| = |n_3| = 1/\sqrt{3}$.

   (b) How many such planes are possible? (The answer to this question tells us why these specially oriented planes are referred to as *octahedral planes*).

8. Consider an octahedral plane as described in the previous question.

   (a) Determine the expression for the normal stress on this plane (referred to to as the octahedral normal stress). Express your answer in terms of stress invariants.

   (b) Determine the expression for the octahedral shear stress i.e. the traction component lying in the same plane that contains the traction vector and the unit outward normal. Express your answer in terms of stress invariants.

(a) $\frac{1}{3}I_1$ (b) $\frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2}$

9. Determine the principal stresses and their directions for each of the sets of stress components in the following. Also calculate the maximum shear stress and the octahedral shear stress. The units of stress are MPa.

   (a) $\sigma_{xx} = 15$, $\sigma_{yy} = -4$, $\sigma_{zz} = 10$, $\sigma_{xy} = -3$, $\sigma_{xz} = 0$, $\sigma_{yz} = 1$.

   (b) $\sigma_{xx} = 10$, $\sigma_{yy} = -5$, $\sigma_{zz} = 0$, $\sigma_{xy} = -5$, $\sigma_{xz} = 0$, $\sigma_{yz} = 0$.

   (c) $\sigma_{xx} = 10$, $\sigma_{yy} = 0$, $\sigma_{zz} = 0$, $\sigma_{xy} = 0$, $\sigma_{xz} = 0$, $\sigma_{yz} = 0$.

   For each of these problems try to determine the solutions using Python or MATLAB. Note that both Python and MATLAB have built-in functions to solve eigenvalue problems.

(a) $\sigma^{(1)} = 15.467$ MPa, $\hat{\mathbf{n}}^{(1)} = \mp\hat{\mathbf{i}}(0.9877) \pm \hat{\mathbf{j}}(0.1537) \pm \hat{\mathbf{k}}(0.0281)$;
$\sigma^{(2)} = 10.063$ Mpa, $\hat{\mathbf{n}}^{(2)} = \pm\hat{\mathbf{i}}(0.0381) \pm \hat{\mathbf{j}}(0.0628) \pm \hat{\mathbf{k}}(0.9973)$;
$\sigma^{(3)} = -4.530$ Mpa, $\hat{\mathbf{n}}^{(3)} = \mp\hat{\mathbf{i}}(0.1515) \mp \hat{\mathbf{j}}(0.9861) \pm \hat{\mathbf{k}}(0.0679)$.
Max. shear stress = 10 MPa;
Octahedral shear stress = 8.45 MPa

(b) This is effectively a 2D situation:
$\sigma_{p1} = 11.514$ MPa, $\hat{\mathbf{n}}^{(1)} = \mp\hat{\mathbf{i}}(0.9571) \pm \hat{\mathbf{j}}(0.2898)$;
$\sigma_{p2} = -6.514$ MPa, $\hat{\mathbf{n}}^{(2)} = \pm\hat{\mathbf{i}}(0.2898) \pm \hat{\mathbf{j}}(0.9571)$;
$\sigma_{p3} = 0$ MPa, $\hat{\mathbf{n}}^{(3)} = \pm\hat{\mathbf{k}}(1)$.
Max. shear stress = 9.014 MPa;
Octahedral shear stress = 7.454 MPa

(b) This is effectively a 1D situation:
$\sigma_{p1} = 10$ MPa, $\hat{\mathbf{n}}^{(1)} = \pm\hat{\mathbf{i}}(1)$ i.e. the original x-axis
$\sigma_{p2} = \sigma_{p3} = 0$ MPa, $\hat{\mathbf{n}}^{(2)} = \pm\hat{\mathbf{i}}(0.2898) \pm \hat{\mathbf{j}}(0.9571)$;
The other two principal directions are any two mutually perpendicular directions perpendicular to $\hat{\mathbf{n}}^{(1)} \equiv x -$ axis
Max. shear stress = 5 MPa;
Octahedral shear stress = 4.714 MPa

10. Referring to coordinate axes aligned along the principal directions and with the principal stresses denoted by σ_{p1}, σ_{p2}, σ_{p3}, we have for a plane characterized by the unit normal $[\hat{\mathbf{n}}] = [n_1\ n_2\ n_3]^\mathsf{T}$:

$$\begin{aligned}(T_{nn})^2 + (T_{ns})^2 &= (\sigma_{p1}n_1)^2 + (\sigma_{p2}n_2)^2 + (\sigma_{p3}n_3)^2,\\ T_{nn} &= \sigma_{p1}n_1^2 + \sigma_{p2}n_2^2 + \sigma_{p3}n_3^2,\\ 1 &= n_1^2 + n_2^2 + n_3^2.\end{aligned}$$

Solve the above system of equations for n_1^2, n_2^2, and n_3^2 to show that:

$$\begin{aligned}n_1^2 &= \frac{(T_{nn} - \sigma_{p2})(T_{nn} - \sigma_{p3}) + T_{ns}^2}{(\sigma_{p1} - \sigma_{p2})(\sigma_{p1} - \sigma_{p3})},\\ n_2^2 &= \frac{(T_{nn} - \sigma_{p3})(T_{nn} - \sigma_{p1}) + T_{ns}^2}{(\sigma_{p2} - \sigma_{p3})(\sigma_{p2} - \sigma_{p1})},\\ n_3^2 &= \frac{(T_{nn} - \sigma_{p1})(T_{nn} - \sigma_{p2}) + T_{ns}^2}{(\sigma_{p3} - \sigma_{p1})(\sigma_{p3} - \sigma_{p2})}.\end{aligned}$$

Take $\sigma_{p1} > \sigma_{p2} > \sigma_{p3}$ and noting that $n_1^2, n_2^2, n_3^2 \geq 0$, show that

$$\begin{aligned}(T_{nn} - \sigma_{p2})(T_{nn} - \sigma_{p3}) + T_{ns}^2 &\geq 0,\\ (T_{nn} - \sigma_{p3})(T_{nn} - \sigma_{p1}) + T_{ns}^2 &\leq 0,\\ (T_{nn} - \sigma_{p1})(T_{nn} - \sigma_{p2}) + T_{ns}^2 &\geq 0.\end{aligned}$$

Recast the above inequalities in the following forms:

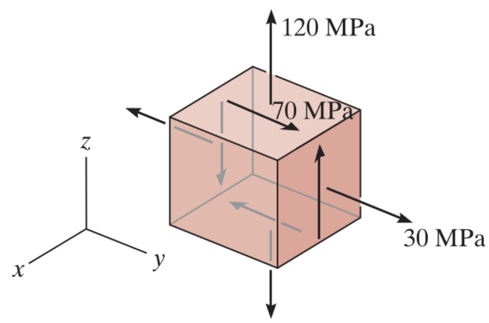
$$\begin{aligned}\left(T_{nn} - \frac{\sigma_{p2} + \sigma_{p3}}{2}\right)^2 + T_{ns}^2 &\geq \left(\frac{\sigma_{p2} - \sigma_{p3}}{2}\right)^2,\\ \left(T_{nn} - \frac{\sigma_{p3} + \sigma_{p1}}{2}\right)^2 + T_{ns}^2 &\leq \left(\frac{\sigma_{p3} - \sigma_{p1}}{2}\right)^2,\\ \left(T_{nn} - \frac{\sigma_{p1} + \sigma_{p2}}{2}\right)^2 + T_{ns}^2 &\geq \left(\frac{\sigma_{p1} - \sigma_{p2}}{2}\right)^2.\end{aligned}$$

Considering the above inequalities to depict regions in a $T_{nn} - T_{ns}$ plane, plot out the boundaries of those regions and the intersection of the valid regions. Note that these boundaries are in the form of circles; they are referred to as the Mohr's circles for a 3D state of stress.

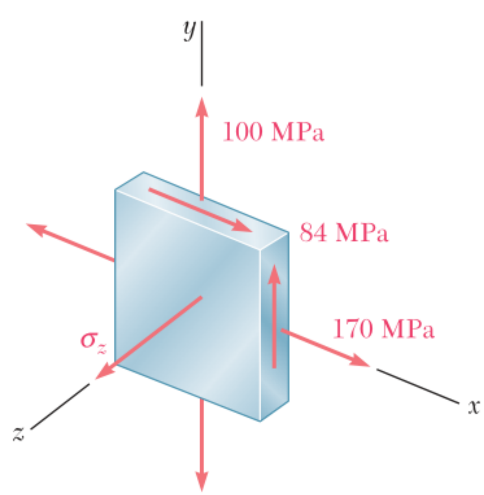
11. Considering the Mohr's circles for a 3D state of stress, as discussed in the previous problem, answer the following:
    (a) What is the maximum shear stress?
    (b) What is the normal stress associated with the maximum shear stress? Think about the opposite situation: what is the shear stress associated with the principal stresses?

Do the next two problems using both the Mohr's circle approach and the algebraic approach. Plot the 3D Mohr's circles using Python or MATLAB.

12. The stress at a point is shown on the element as shown in the figure. Determine the principal stresses and the absolute maximum shear stress. [$\sigma_{\text{max}} = 158$ MPa, $\sigma_{\text{min}} = -8.22$ MPa, $\sigma_{\text{intermediate}} = 0$ MPa, $\tau_{\text{max,abs}} = 83.22$ MPa]



13. For the state of stress, determine the maximum shearing stress when (a) $\sigma_z = 0$, (b) $\sigma_z = 60$ MPa, (c) $\sigma_z = -60$ MPa. [(a) 113.0 MPa, (b) 91.0 MPa, (c) 143.0 MPa]



14. (*Question for thinking!*) It was shown in class that the plane on which the maximum in-plane shear stress occurs is inclined to the planes on which the principal stress by 45°.

    (a) It is known that ductile materials usually fail due to shear. From this, explain the *cup-and-cone* nature of the failure surface in a uniaxial tensile test of a ductile specimen.

    (b) It is known that brittle materials usually fail due to normal stresses. From this, explain the failure of a chalk piece (using for writing on blackboards) along a helical surface when we apply torsion to its two ends.