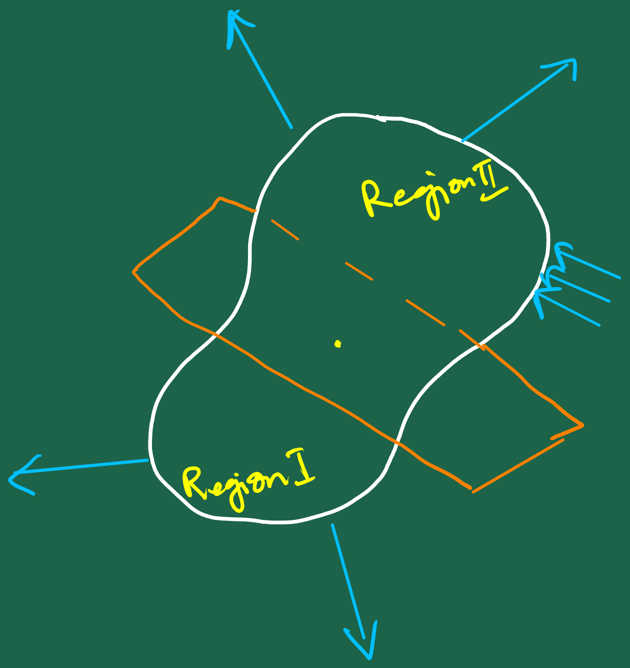
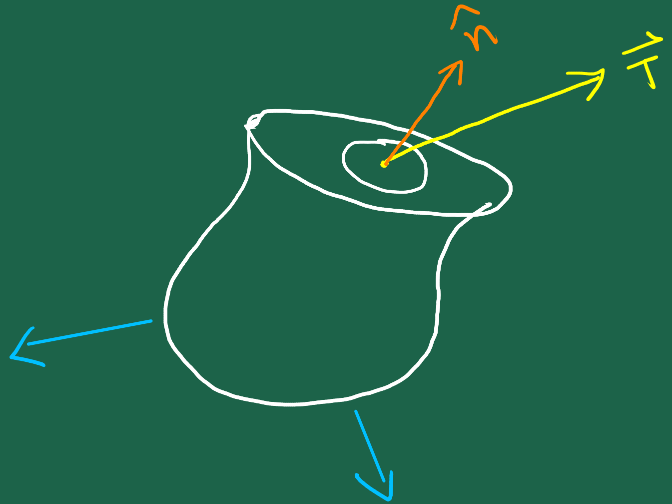
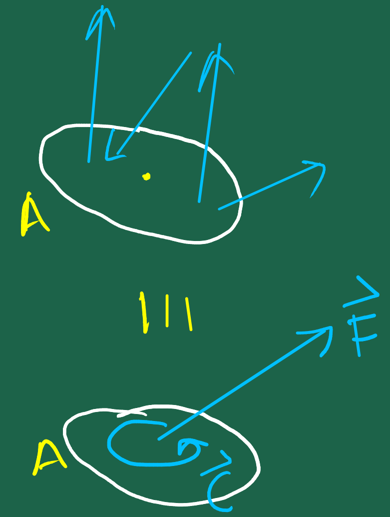
Concept of Traction







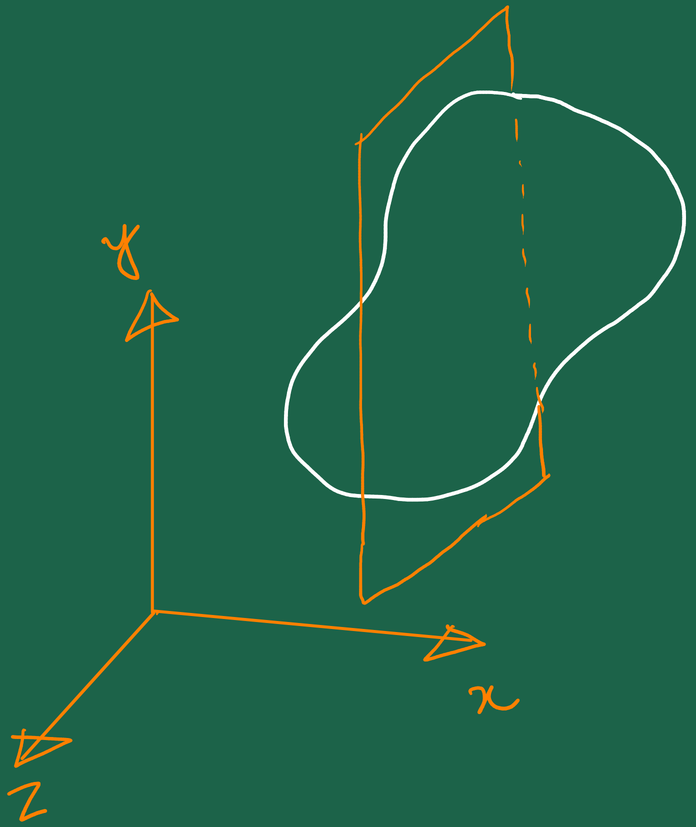
$A \to 0 \quad \to \quad |\vec{C}| \to 0$

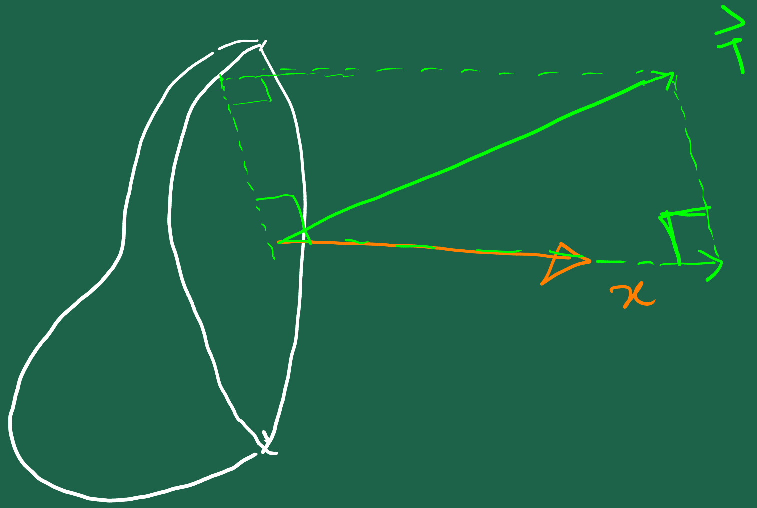
$|\vec{F}| \to$ finite value

$$\vec{T} = \lim_{A \to 0} \frac{\vec{F}}{A}$$

Traction

$$\vec{T} \equiv \vec{T}(\vec{x}, \hat{n})$$



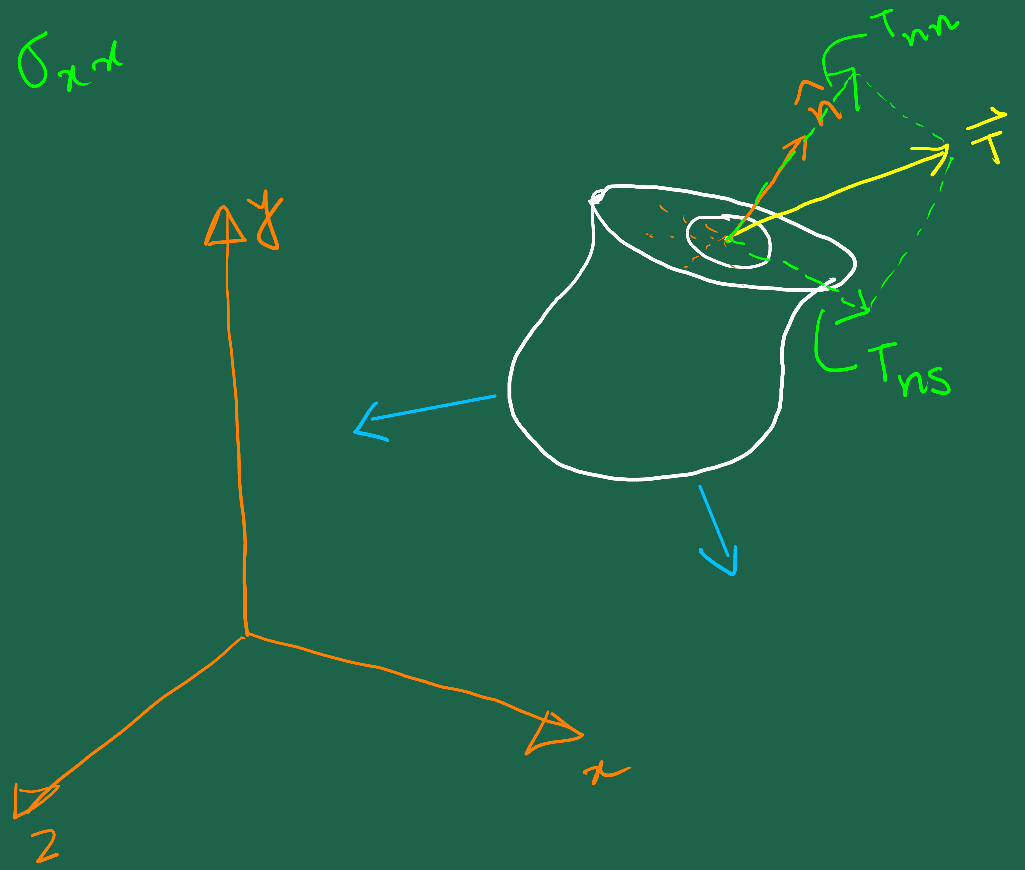


$$\sigma_{xx} = \vec{T} \cdot \hat{i}$$

$$\tau_{xy} = \vec{T} \cdot \hat{j}$$

$$\tau_{xz} = \vec{T} \cdot \hat{k}$$

$$[\underset{\sim}{\sigma}] = \underbrace{\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}}_{\text{Stress tensor}}$$



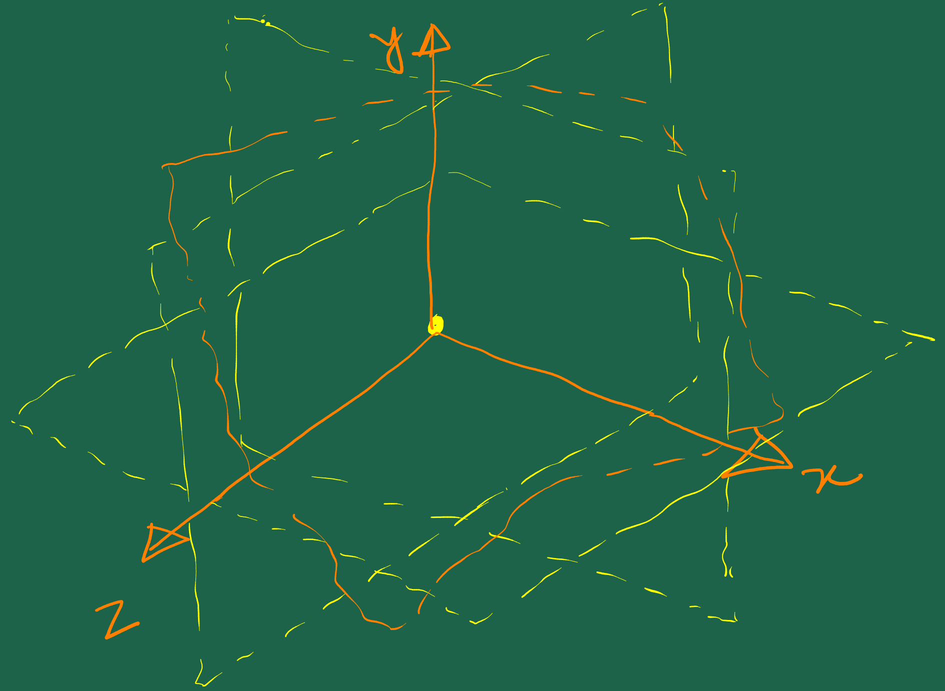
$\vec{T}$ can be decomposed into two ways:

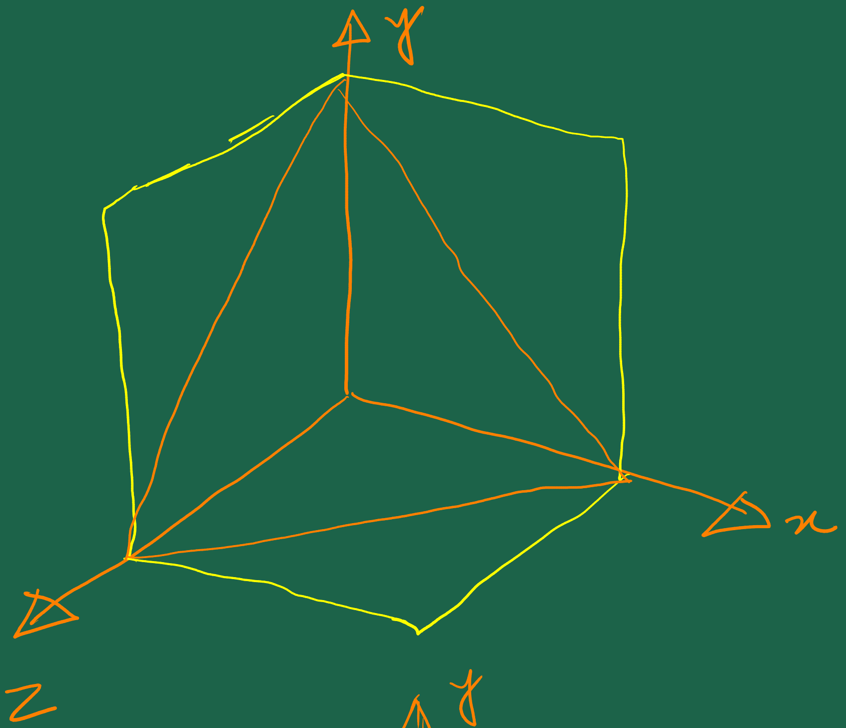
$\vec{T} = T_x\hat{i} + T_y\hat{j} + T_z\hat{k}$

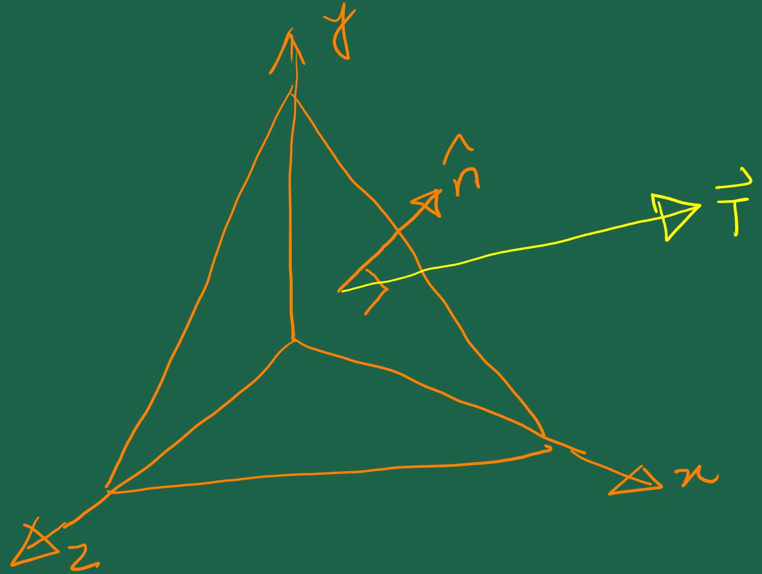
$\vec{T} = T_{nn}\hat{n} + T_{ns}\hat{e}_s$

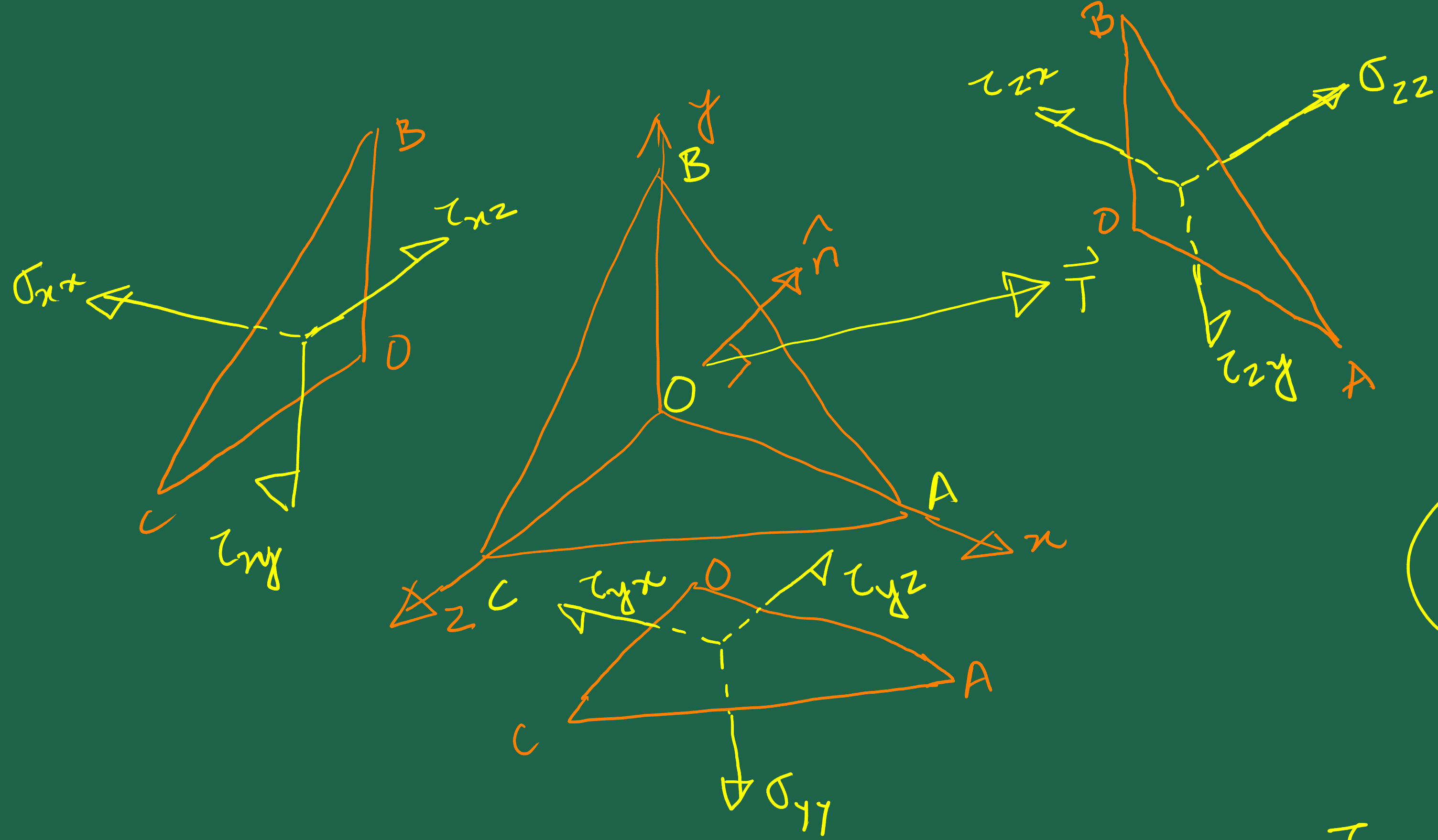
$\hat{e}_s \perp \hat{n}$ & $\hat{e}_s$ is lying in the same plane containing $\vec{T}$ & $\hat{n}$

Cauchy









$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$$

$$\left\{\begin{array}{l} \Delta OCB = \Delta ABC\, n_x \\ \Delta OAC = \Delta ABC\, n_y \\ \Delta OBA = \Delta ABC\, n_z \end{array}\right.$$

$\sum F_x = 0$

$$T_x\, \Delta ABC - \sigma_{xx}\, \Delta OCB - \tau_{yx}\, \Delta OAC - \tau_{zx}\, \Delta OBA = 0$$

$$\Rightarrow \quad T_x\, \Delta ABC$$

$$\Rightarrow \quad T_x = \sigma_{xx} n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

$$\sum F_y = 0 \Rightarrow \quad T_y = \tau_{xy} n_x + \sigma_{yy} n_y + \tau_{zy} n_z$$

$$\sum F_z = 0 \Rightarrow \quad T_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_{zz} n_z$$

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

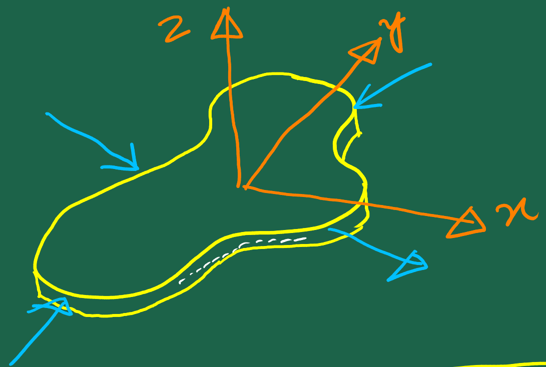
$$\boxed{\left[\vec{T}\right] = \left[\underset{\approx}{\sigma}\right]^T \left[\hat{n}\right]}$$

T_{nn} is the component of $\vec{T}$ along $\hat{n}$ itself

$$T_{nn} = \vec{T} \cdot \hat{n}$$
$$= [\vec{T}]^T [\hat{n}]$$
$$= \left([\underset{=}{\sigma}]^T [\hat{n}] \right)^T [\hat{n}]$$
$$= [\hat{n}]^T [\underset{=}{\sigma}] [\hat{n}] \rightarrow \text{quadratic form}$$

$$(T_x \hat{i} + T_y \hat{j} + T_z \hat{k}) \cdot (n_x \hat{i} + n_y \hat{j} + n_z \hat{k})$$
$$= T_x n_x + T_y n_y + T_z n_z$$

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$



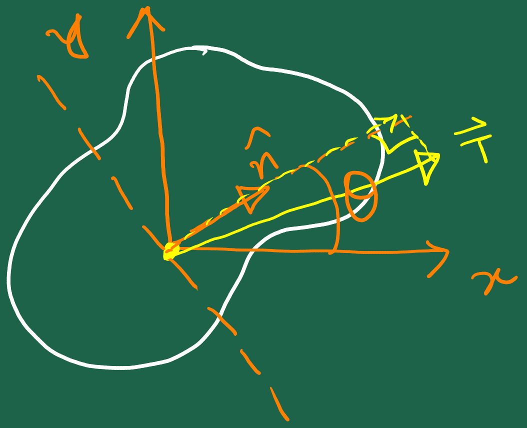
Plane Stress

$$\sigma_{zz} = \tau_{zy} = \tau_{zx} = 0$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left([A][B] \right)^T = [B]^T [A]^T$$

$$[x]^T [A] [x]$$



$$\hat{n} = n_x\hat{i} + n_y\hat{j} + \cancel{n_z}\hat{k}$$

$\cos 90^\circ = 0$

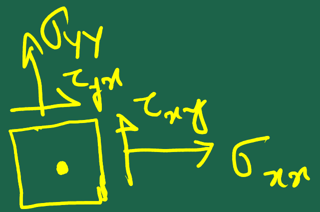
$$= n_x\hat{i} + n_y\hat{j}$$

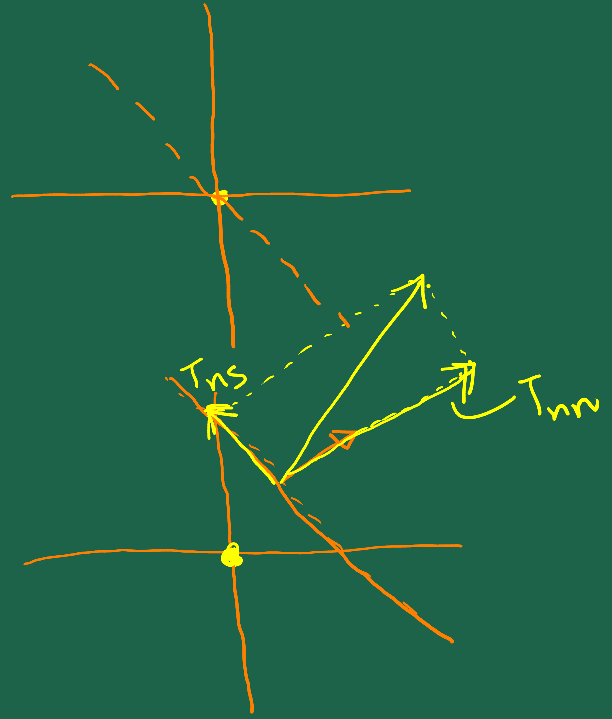
$$= \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \rightarrow [\hat{n}] = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

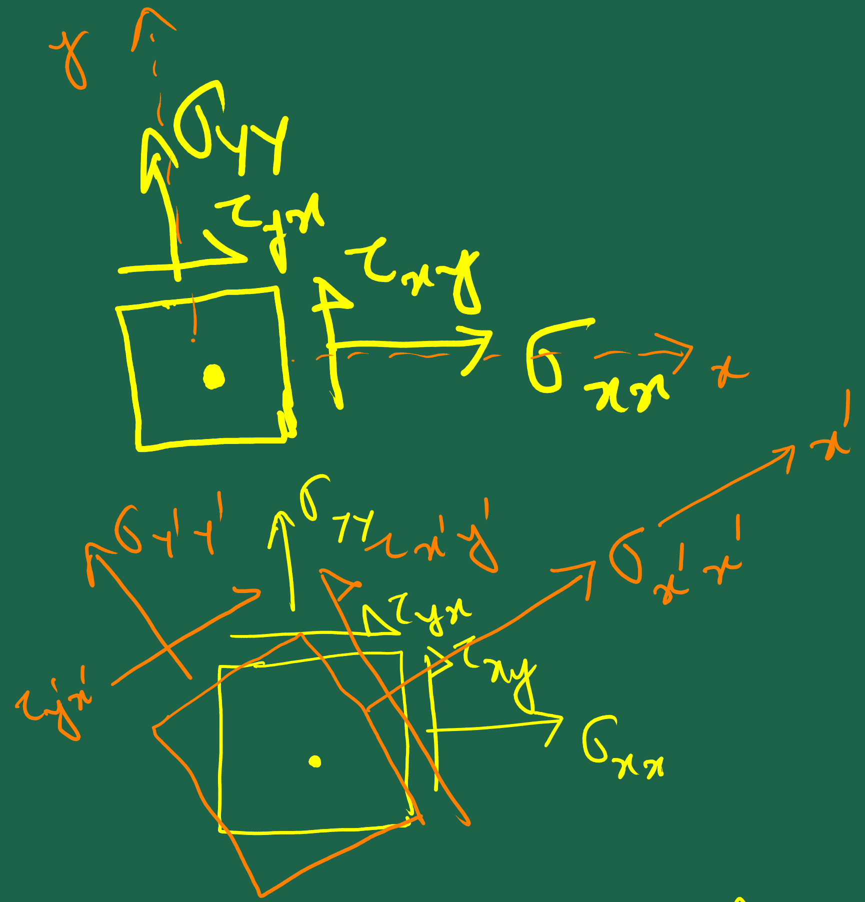
$$T_{nn} = [\hat{n}]^T[\underset{\sim}{\sigma}][\hat{n}]$$

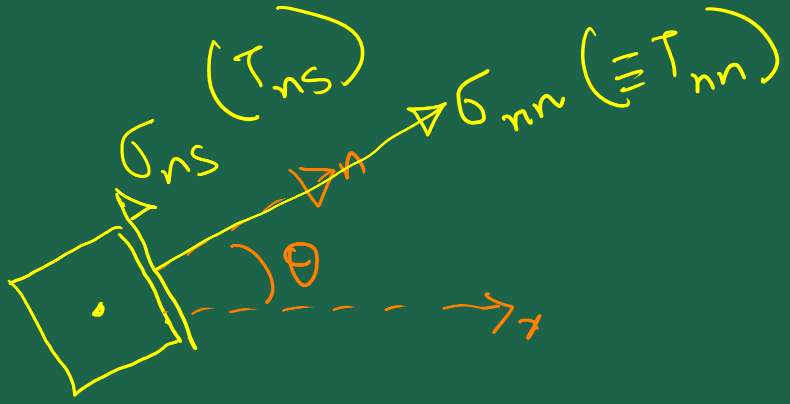
$$= \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{xy} & \sigma_{yy}\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$= \sigma_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$









$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$

$\sigma_{x'x'}$

$$(\sigma)(\hat{n}) \leftarrow \vec{T} = T_{nn}\,\hat{n} + T_{ns}\,\hat{e}_s$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = T_{nn}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + T_{ns}\begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{xy} & \sigma_{yy}\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = T_{nn}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + T_{ns}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta = T_{nn}\cos\theta - T_{ns}\sin\theta \quad \text{— (1)}$$

$$\tau_{xy}\cos\theta + \sigma_{yy}\sin\theta = T_{nn}\sin\theta + T_{ns}\cos\theta \quad \text{— (2)}$$

Eliminate T_{nn} and obtain T_{ns} expression

$$(2)\times\cos\theta \;-\; (1)\times\sin\theta$$

$$T_{ns} = \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) + \left(\sigma_{yy}-\sigma_{xx}\right)\cos\theta\sin\theta$$

$\tau_{x'y'}$

$$[\hat{e}_s] = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

OR

$$\begin{bmatrix}\sin\theta\\ -\cos\theta\end{bmatrix}$$

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$

$$\tau_{x'y'} = \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) + \left(\sigma_{yy}-\sigma_{xx}\right)\cos\theta\sin\theta$$

For $\sigma_{y'y'}$, just replace θ by $\theta + 90^\circ$ in the exp. of $\sigma_{x'x'}$

$$\sigma_{x'x'} = \frac{1}{2}\sigma_{xx}(1+\cos 2\theta) + \tau_{xy}\sin 2\theta + \frac{1}{2}\sigma_{yy}(1-\cos 2\theta)$$

$$\sigma_{x'x'} = \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \tau_{xy}\cos 2\theta - \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\theta$$

$$\sigma_{x'x'} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) = \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \quad — \quad \#_1$$

$$\tau_{x'y'} = \tau_{xy}\cos 2\theta - \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta \quad — \quad \#_2$$

$$(\#_1)^2 + (\#_2)^2$$

$$\Rightarrow \left\{\sigma_{x'x'} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\right\}^2 + \tau^2_{x'y'} = \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2\cos^2 2\theta + \tau^2_{xy}\sin^2 2\theta$$
$$+ \cancel{(\sigma_{xx} - \sigma_{yy})\cos 2\theta\, \tau_{xy}\sin 2\theta}$$
$$+ \tau^2_{xy}\cos^2 2\theta + \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2\sin^2 2\theta$$
$$\cancel{- \tau_{xy}\cos 2\theta(\sigma_{xx} - \sigma_{yy})\sin 2\theta}$$

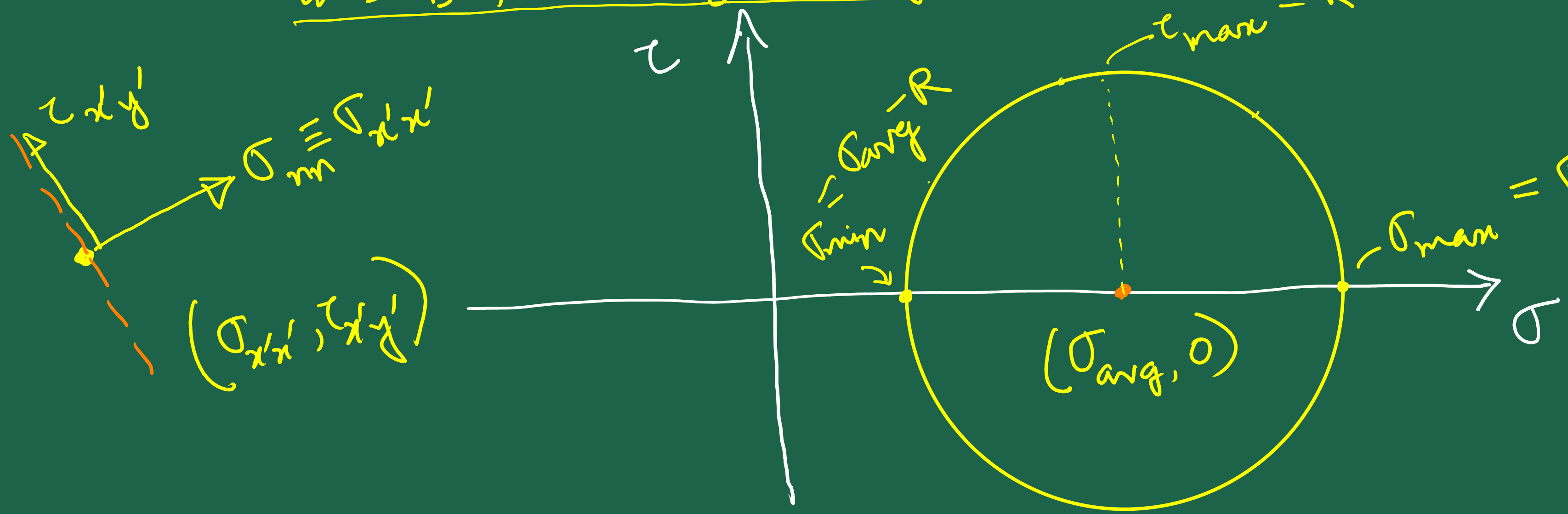
$$\Rightarrow \left\{\sigma_{x'x'} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\right\}^2 + \tau^2_{x'y'} = \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau^2_{xy}$$

$$\left\{\sigma_{x'x'} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\right\}^2 + \tau_{x'y'}^2 = \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2$$

$$\Rightarrow \quad (\sigma_{x'x'} - \sigma_{avg})^2 + \tau_{x'y'}^2 = R^2,$$

where $\sigma_{avg} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$ and $R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$

This is in the form of a circle!



Otto

Mohr's Circle

Principal stresses

$\leftarrow$ very special plane where $\vec{T} \parallel \hat{n}$

$$\vec{T} = \cancel{T_{nn}}^{\,\sigma_p}\,\hat{n} + \cancel{T_{ns}\,\hat{e}_s}^{\,0}$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta_p \\ \sin\theta_p \end{bmatrix} = \sigma_p \begin{bmatrix} \cos\theta_p \\ \sin\theta_p \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_{xx} - \sigma_p & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \sigma_p \end{bmatrix} \begin{bmatrix} \cos\theta_p \\ \sin\theta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} \sigma_{xx}-\sigma_p & \tau_{xy} \\ \tau_{xy} & \sigma_{yy}-\sigma_p \end{vmatrix} = 0$$

$$\Rightarrow \quad \sigma_p = \frac{(\sigma_{xx}+\sigma_{yy}) \pm \sqrt{(\sigma_{xx}-\sigma_{yy})^2 + 4\tau_{xy}^2}}{2}$$

$$= \underbrace{\frac{\sigma_{xx}+\sigma_{yy}}{2}}_{\sigma_{avg}} \pm \underbrace{\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}}_{R}$$

$$= \sigma_{avg} \pm R$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta_p \\ \sin\theta_p \end{bmatrix} = \sigma_p \begin{bmatrix} \cos\theta_p \\ \sin\theta_p \end{bmatrix}$$

$$\sigma_{xx}\cos\theta_p + \tau_{xy}\sin\theta_p = \sigma_p\cos\theta_p \quad \text{--- } (\#_3)$$

$$\tau_{xy}\cos\theta_p + \sigma_{yy}\sin\theta_p = \sigma_p\sin\theta_p \quad \text{--- } (\#_4)$$

$$(\#_4) \div (\#)_3$$

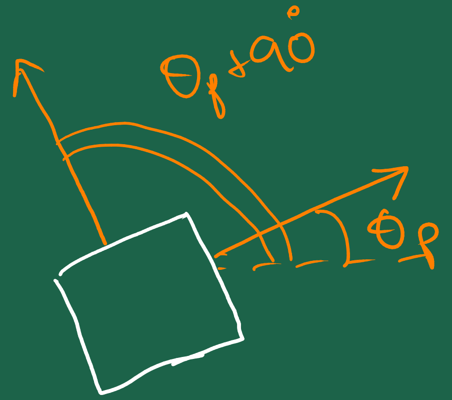
$$\Rightarrow \quad \frac{\tau_{xy}\cancel{\cos\theta_p} + \sigma_{yy}\cancel{\sin\theta_p}^{\tan\theta_p}}{\sigma_{xx}\cancel{\cos\theta_p} + \tau_{xy}\cancel{\sin\theta_p}^{\tan\theta_p}} = \tan\theta_p$$

$$\Rightarrow \quad \tau_{xy} + \sigma_{yy} \tan\theta_p = \sigma_{xx} \tan\theta_p + \tau_{xy} \tan^2\theta_p$$

$$\Rightarrow \quad \tau_{xy}\left(1 - \tan^2\theta_p\right) = \left(\sigma_{xx} - \sigma_{yy}\right)\tan\theta_p$$

$$\Rightarrow \quad \frac{2\tan\theta_p}{1 - \tan^2\theta_p} = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

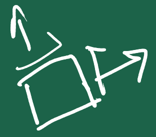
$$\Rightarrow \quad \tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$



$$2\theta_p \quad , \quad 2\theta_p + 180^\circ$$

↳ Principal planes : θ_p , $\theta_p + 90^\circ$

Max. magnitude of $\tau_{x'y'}$ is R. For the orientation of the plane : $\rightarrow \theta_s$



$$\tau_{x'y'} = \tau_{xy} \cos 2\theta - \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right) \sin 2\theta$$

$$\frac{d\,\tau_{x'y'}}{d\theta} = 0$$

$$\Rightarrow \quad \tan 2\theta_s = \frac{\sigma_{yy} - \sigma_{xx}}{2\tau_{xy}}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

$\rightarrow \tan 2\theta_p \tan 2\theta_s = -1$

↳ Indicates that the planes on which the principal stresses occur and the ones on which the max τ occur are at an angle of 45°

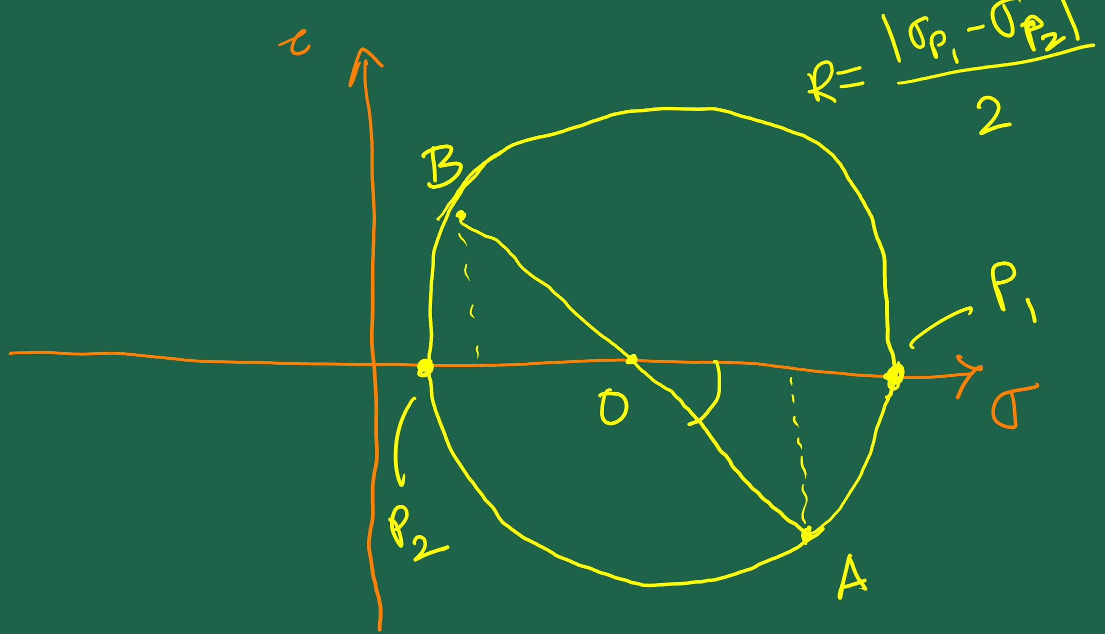
→ Maximum in-plane shear stress

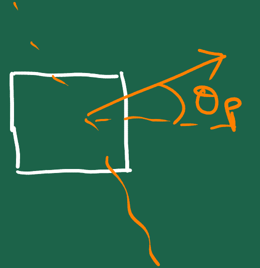
CCW $-$

CW $+$

$$A \begin{cases} \sigma_{xx} = 20\,\text{MPa} \\ \tau_{xy} = 10\,\text{MPa} \end{cases}$$

$$B \begin{cases} \sigma_{yy} = 5\,\text{MPa} \\ \tau_{yx} = 10\,\text{MPa} \end{cases}$$

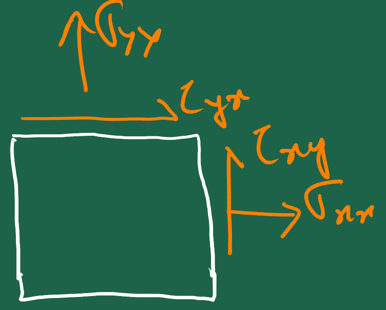


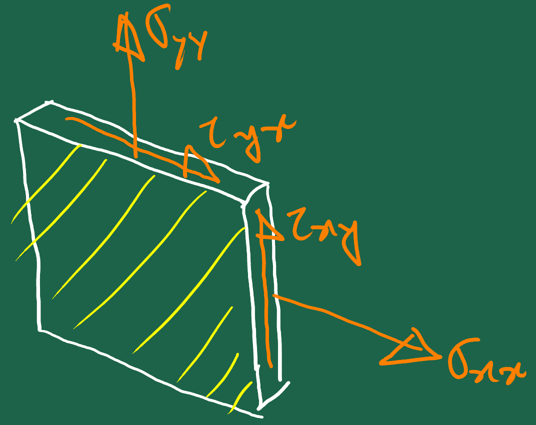


$$\tan \angle AOP_1 = \frac{\tau_{xy}}{\frac{1}{2}(\sigma_{xx} - \sigma_{yy})} = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

Go back and check that

$$\tan \angle AOP_1 = \tan 2\theta_p$$





In general,

Absolute max value of the shear stress

$\neq$ Max in-plane shear stress

Max. in-plane shear stress $= R = \dfrac{|\sigma_{p_1} - \sigma_{p_2}|}{2}$

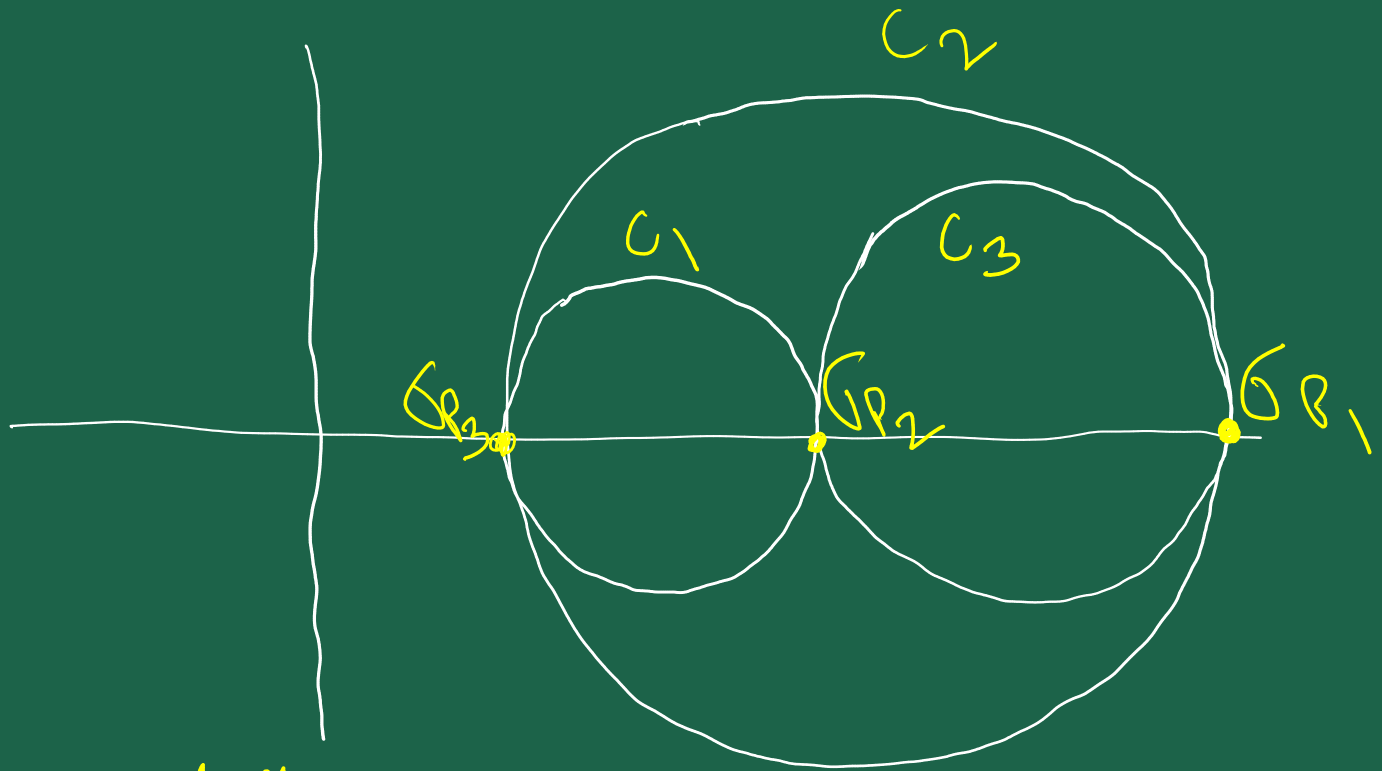
Note: $\sigma_{zz}=0$, $\tau_{zx}=0$ ✓, $\tau_{yz}=0$ ✓

$\downarrow$

σ_{p_3}

Say $\sigma_{p_1} > \sigma_{p_2} > 0$ and we found that $\sigma_{p_3} = 0$ $\Big\} \rightarrow$ $\therefore$ Max value of shear stress

$= \frac{1}{2}(\sigma_{p_1} - \underbrace{\cancel{\sigma_{p_3}}}_{0}) = \frac{1}{2}\sigma_{p_1}$



$\sigma_{p_1} > \sigma_{p_2} > \sigma_{p_3}$

Max in-plane shear stress

C_1 : $\dfrac{\sigma_{p_2} - \sigma_{p_3}}{2}$

C_2 : $\dfrac{\sigma_{p_1} - \sigma_{p_3}}{2}$

C_3 : $\dfrac{\sigma_{p_1} - \sigma_{p_2}}{2}$

max : $\dfrac{\sigma_{p_1} - \sigma_{p_3}}{2}$