

Energy Methods

Strain Energy

1st law of Thermodynamics :

$$W + \cancel{Q} = \Delta U + \cancel{\Delta K}$$

Adiabatic : $Q = 0$

Static equilibrium : $\Delta K = 0$

$$W = \Delta U$$

Mechanical system is said to be elastic if the internal forces are conservative.

If forces are conservative then some kind of potential energy will be associated with it.

∴ The ^{conservative} internal forces in an elastic mechanical must also have a potential energy associated.

↳ strain energy

The strain energy differs from the internal energy only by an additive constant

$$W = \Delta U$$

↳ change in strain energy

$$\Delta \bar{U} = \Delta U$$

$$\bar{U} = U + C$$

Consider simple cases:



$$\begin{aligned} W &= \int_0^{\Delta} \bar{F} dx \\ &= \int_0^{\Delta} P \frac{x}{\Delta} dx \end{aligned}$$

Assume force is linearly related to the deformation

$$F = P \frac{x}{\Delta}$$

$$\begin{aligned} W &= \int_V \vec{\sigma} \cdot \vec{u} dV \\ &\quad + \int_S \vec{T} \cdot \vec{u} dS \end{aligned}$$

$$W = \frac{P}{\Delta} \frac{\Delta^2}{2} = \frac{1}{2} P \Delta = \Delta U$$

Due to normal stresses

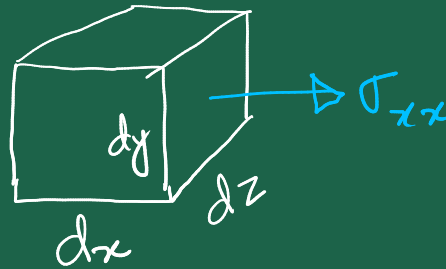
$$dU = \frac{1}{2} \underbrace{\sigma_{xx} dy dz}_{\text{force}} \underbrace{\epsilon_{xx} dx}_{\text{displ.}}$$

$$= \frac{1}{2} \sigma_{xx} \epsilon_{xx} dx dy dz$$

$$= \frac{1}{2} \sigma_{xx} \epsilon_{xx} dV$$

$$U_0 = \frac{dU}{dV} = \frac{1}{2} \sigma_{xx} \epsilon_{xx}$$

↳ Strain energy density



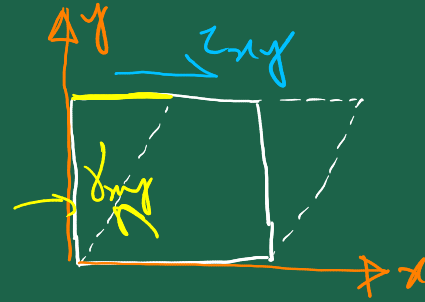
Due to shear stress

$$dU = \frac{1}{2} \underbrace{\tau_{xy} dx dz}_{\text{force}} \underbrace{\delta_{xy} dy}_{\text{displ.}}$$

$$= \frac{1}{2} \tau_{xy} \delta_{xy} dx dy dz$$

$$= \frac{1}{2} \tau_{xy} \delta_{xy} dV$$

$$U_0 = \frac{dU}{dV} = \frac{1}{2} \tau_{xy} \delta_{xy}$$



Strain energy for various types of loading

Axial load

$$\begin{aligned}\Delta U &= \int dU = \int \frac{dU}{dV} dV = \int U_0 dV \\ &= \int \frac{1}{2} \sigma_{xx} \epsilon_{xx} dV \\ &= \int \frac{\sigma_{xx}^2}{2E} dV\end{aligned}$$

$$= \int \frac{N^2}{2AE} dV = \int_0^L \frac{N^2}{2EA^2} A dx = \int_0^L \frac{N^2}{2AE} dx$$

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E} \left[\sigma_{xx} - \nu \left(\sigma_{yy} + \sigma_{zz} \right) \right] \\ &= \frac{\sigma_{xx}}{E}\end{aligned}$$

Bending moment

$$\Delta U = \int \frac{1}{2} \sigma_{xx} \epsilon_{xx} dV = \int \frac{\sigma_{xx}^2}{2E} dV$$

$$= \int \frac{\tilde{M} \tilde{y}}{2EI^2} dV = \int_0^L \int_A \frac{\tilde{M} \tilde{y}}{2EI^2} dA dx$$

$$= \int_0^L \frac{\tilde{M}}{2EI} \left(\int_A \tilde{y} dA \right) dx$$

$$= \int_0^L \frac{\tilde{M}}{2EI^2} I dx$$

$$= \int_0^L \frac{\tilde{M}}{2EI} dx$$

Transverse shear

$$\Delta U = \int \frac{1}{2} \tau_{xy} \gamma_{xy} dV$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$= \int \frac{\tau_{xy}^2}{2G} dV$$

$$\tau_{xy} = \frac{VQ}{It}$$

$$= \int \frac{\tilde{V}^2 Q^2}{2GI^2 t^2} dV$$

$$= \int_0^L \int_A \frac{\tilde{V}^2 Q^2}{2GI^2 t^2} dA dx$$

$$= \int_0^L \frac{\tilde{V}^2}{2GI^2} \left(\int_A \frac{Q^2}{t^2} dA \right) dx$$

Define a form factor for shear

$$f = \frac{A}{I^2} \int_A \frac{Q^2}{t^2} dA$$

$$\Rightarrow \int \frac{Q^2}{t^2} dA = f \frac{I^2}{A}$$

$$\therefore \Delta U = \int_0^L \frac{\vec{V}}{2GI^2} \int \frac{I^2}{A} dx = \int_0^L f \frac{\vec{V}}{2GA} dx$$

For a rectangular cs: $f = \frac{6}{5}$ (check yourself)

Torsion

$$\Delta U = \int \frac{1}{2} \tau_{z\theta} \gamma_{z\theta} dV$$

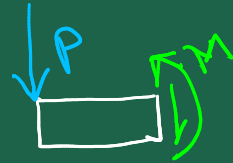
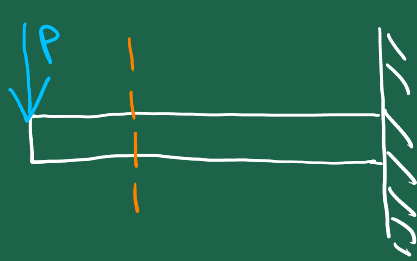
$$\tau_{z\theta} = \frac{T\gamma}{J}$$

$$= \int \frac{\tau_{z\theta}}{2G} dV$$

$$= \int_0^L \int_0^R \frac{\tau \gamma}{2GI^2} 2\pi r dr dz = \int_0^L \frac{T}{2GI^2} \left(\int_0^R 2\pi r^3 dr \right) dz = \int_0^L \frac{T}{2GI^2} \left(\pi \underbrace{\frac{R^4}{2}}_J \right) dz$$

$$= \int_0^L \frac{T}{2GI} dz$$

Limitations of conservation of energy principle



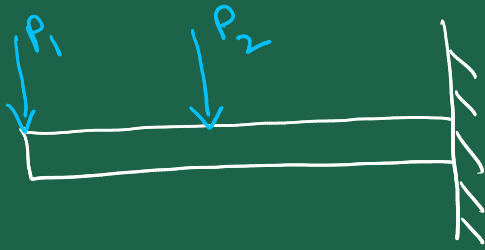
$$M = -Px$$

$$\frac{1}{2} P \Delta = \int_0^L \frac{M^2}{2EI} dx$$

$$= \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{2EI} \int_0^L x^2 dx = \frac{P^2 L^3}{6EI}$$

$$\therefore \Delta = \frac{PL^3}{3EI}$$



$$\frac{1}{2} P_1 \Delta_1 + \frac{1}{2} P_2 \Delta_2 = \int_0^L \frac{M \bar{M}}{2EI} dx$$

We need another eqn to solve for Δ_1 & Δ_2



Principle of virtual work

$$W_e = \Delta U = -W_i$$

$$W = -\Delta U$$

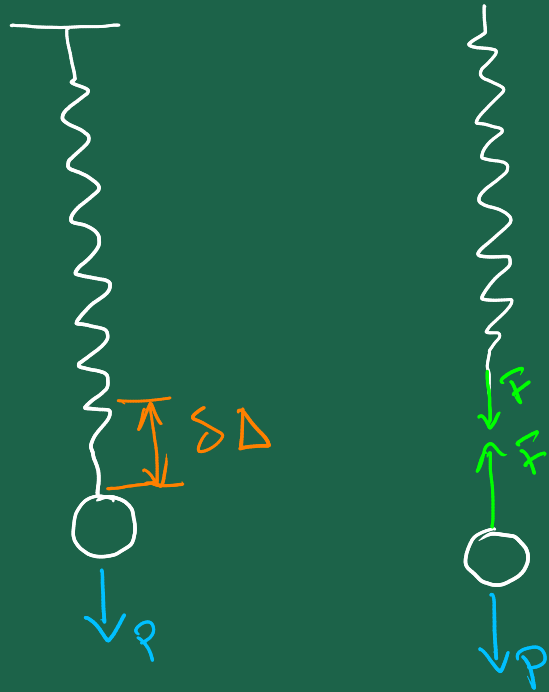
$$W_e = -W_i$$

$\Rightarrow W_e + W_i = 0 \rightarrow$ more general
applicable even to non-elastic systems

Virtual work $\begin{cases} \rightarrow \text{Virtual displacements} \\ \rightarrow \text{Virtual forces} \end{cases}$

Virtual work based on virtual displacements

$$W = \vec{F} \cdot \vec{\delta}$$



$$\delta W_e = -\delta W_i$$

$$\Rightarrow P \delta \Delta = -(-F \delta \Delta)$$

$$\Rightarrow P \delta \Delta = F \delta \Delta$$

$$\Rightarrow (P - F) \delta \Delta = 0$$

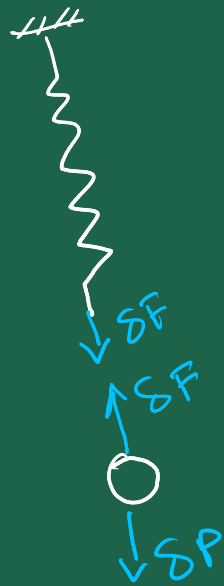
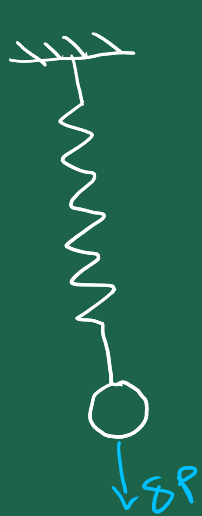
$$\Rightarrow P - F = 0$$

$$\Rightarrow P = F \quad (\text{eqn of static eqb.})$$

$$\delta W_e = -\delta W_i = \delta W_{ie}$$

$$P \delta F = F \delta \Delta$$

Virtual work based on virtual force



$$\delta W_e = -\delta W_i$$

$$\Rightarrow \delta P \Delta = -(-\delta F \Delta_{sp})$$

$$\Rightarrow \delta P \Delta = \delta F \Delta_{sp}$$

We know a priori that $\delta P = \delta F$

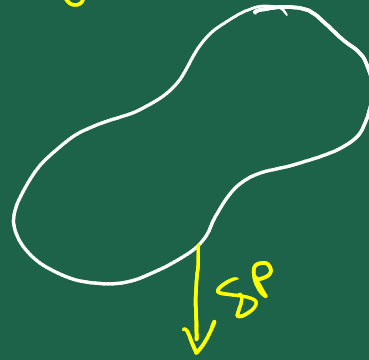
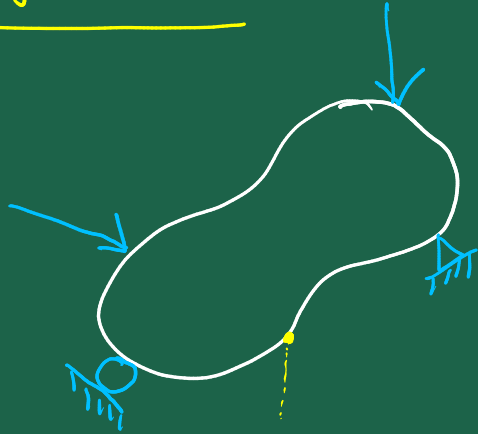
$$\therefore \Delta = \Delta_{sp} \text{ (eqn of compatibility)}$$

$$\delta W_e = -\delta W_i = \delta W_{ie}$$

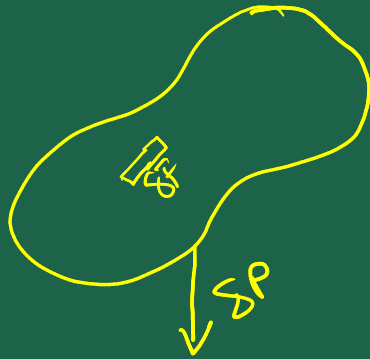
Popov

Virtual work associated with virtual forces as a method for finding deflection

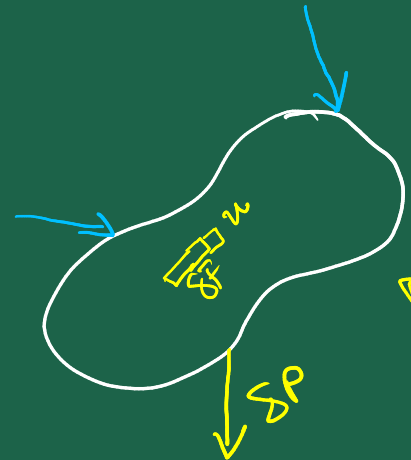
1. Consider the unloaded body and apply a virtual force of magnitude $\delta P = 1$ along the desired dirⁿ



2. Due to δP , internal forces generated all throughout the body. If the body is statically determinate, we can determine them.



3. With δP still applied bring back the actual loading.
Actual loading $\rightarrow$ real deformations $\rightarrow$ Calc δ

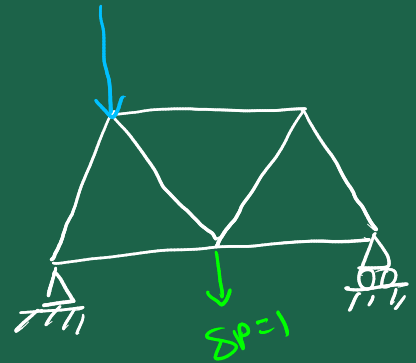


4. Write the ext. virtual work by multiplying δP with Δ , i.e. the deformation along the desired dirⁿ.
- Write the int. virtual work due to all the internal virtual forces multiplied with the deformations at those points.

$$\delta P \times \Delta = \sum \delta F \times u$$

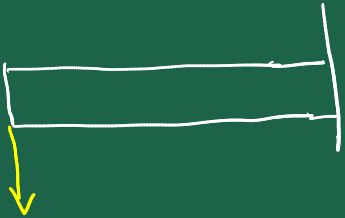
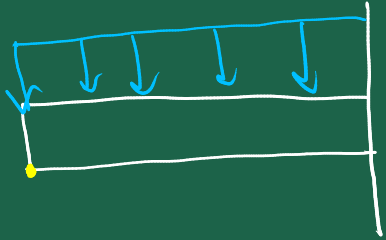
Truss

$$1 \times \Delta = \sum_{i=1}^n \delta F_i \frac{N_i L_i}{A_i E_i}$$



Beam

$$I \times \Delta = \int_0^L m d\theta$$
$$= \int_0^L m \frac{M}{EI} dx$$

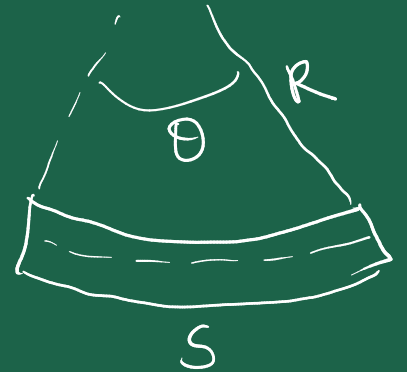


$$MR = EI$$

$$\Rightarrow R = \frac{EI}{M}$$

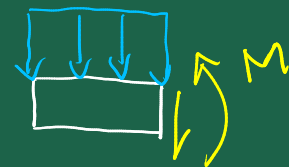
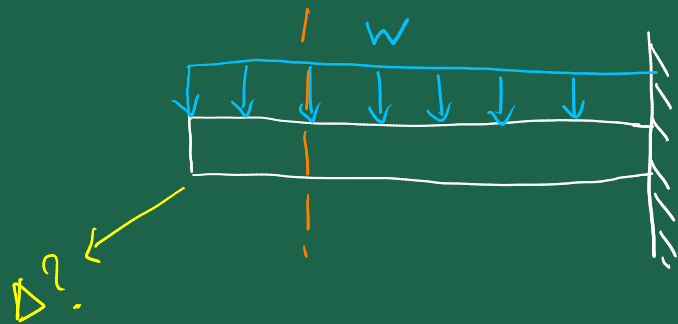
$$\Rightarrow \frac{1}{R} = \frac{M}{EI}$$

$$dx \leftarrow S = R \theta \rightarrow d\theta$$

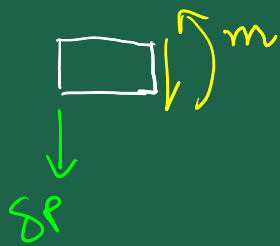
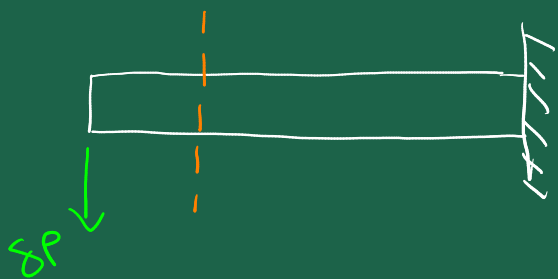


$$d\theta = \frac{S}{R} \leftrightarrow \frac{M}{EI} dx$$

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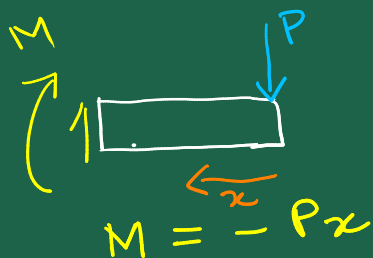
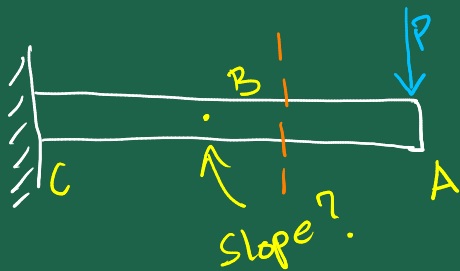
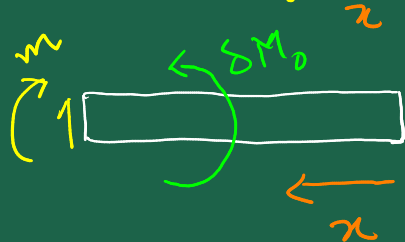
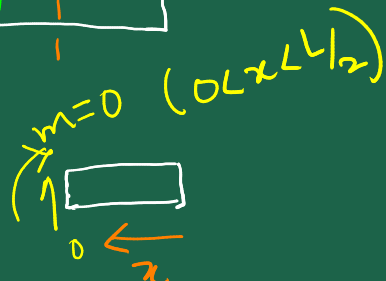
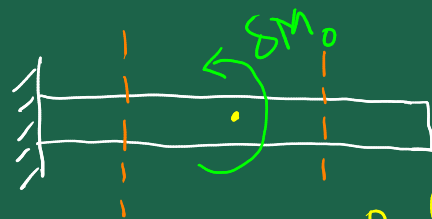
$$M = -\frac{wx^2}{2}$$



$$m = -\cancel{\delta P} x$$

$$\begin{aligned} \cancel{L} \delta P \times \Delta &= \int_0^L m \frac{M dx}{EI} \\ &= \int_0^L (-x) \frac{(-wx^2/2)}{EI} dx \\ &= \frac{w}{2EI} \int_0^L x^3 dx \\ &= \frac{wL^4}{8EI} \end{aligned}$$

#

F Δ M θ 

$$m - \delta M_0 = 0 \Rightarrow m = \delta M_0$$

(L/2 < x < L)

$$\begin{aligned} \delta M_0 \times \theta &= \int_0^L m \frac{M dx}{EI} \\ &= \int_0^{L/2} (0) \frac{(-Px)}{EI} dx + \int_{L/2}^L (1) \frac{(-Px)}{EI} dx \\ &= 0 + \frac{(-P)}{EI} \int_{L/2}^L x dx \\ &= \frac{(-P)}{EI} \frac{3L^2}{8} \\ &= \frac{-3PL^2}{8EI} \end{aligned}$$

CASTIGLIANO'S THEOREMS

$$\Delta = \frac{\partial U}{\partial P}$$

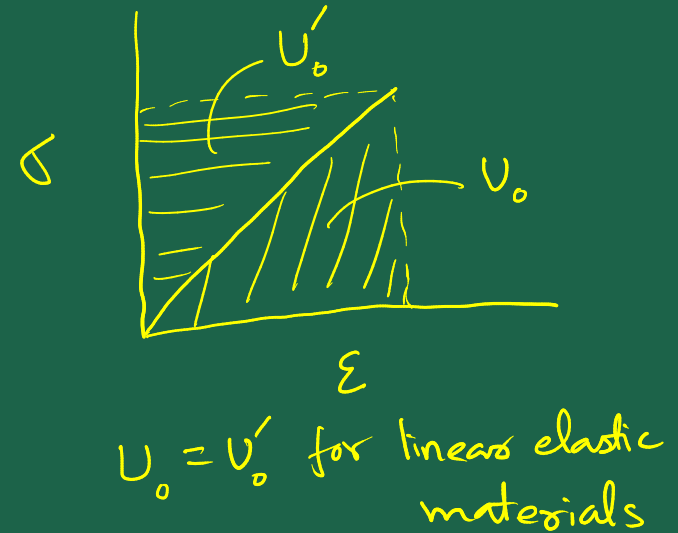
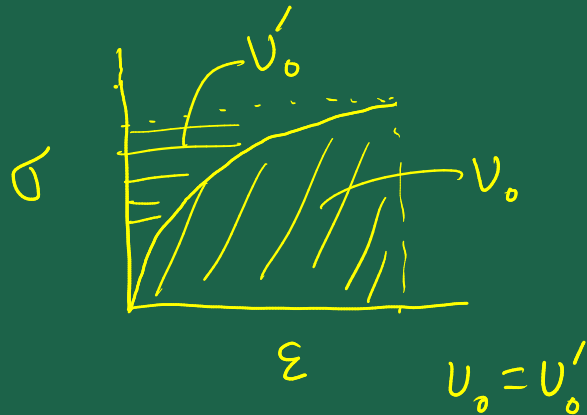
$$\Theta = \frac{\partial U}{\partial M}$$

Castigliano's 2nd theorem
or

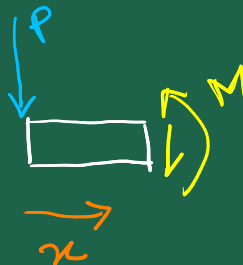
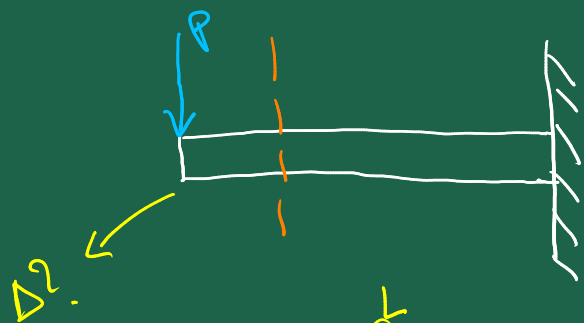
Castigliano's theorem on deflection

Applicable only
to linear elastic
materials

Generalization: $\Delta = \frac{\partial U'}{\partial P}$
 $\Theta = \frac{\partial U'}{\partial M}$ $\rightarrow U'$: Complementary strain energy



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$$M = -Px$$

$$U = \int_0^L \frac{M^2}{2EI} dx$$

$$\Delta = \frac{\partial U}{\partial P}$$

$$= \frac{\partial}{\partial P} \int_0^L \frac{M^2}{2EI} dx$$

$$= \int_0^L \frac{2M \frac{\partial M}{\partial P}}{2EI} dx$$

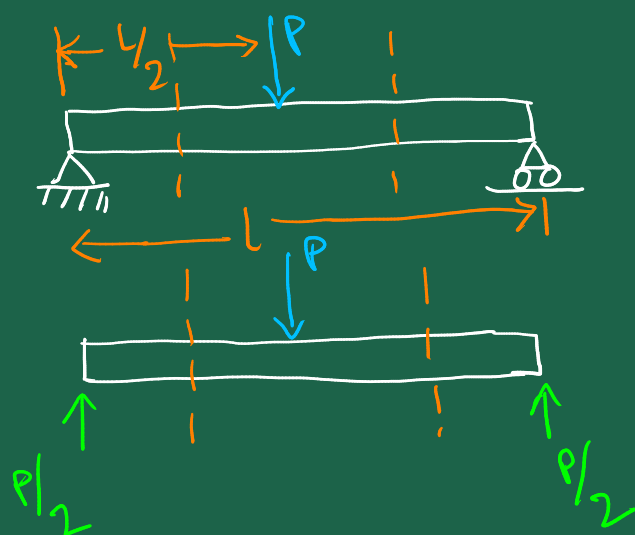
$$= \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dx$$

$$= \int_0^L \frac{(-Px)}{EI} (-x) dx$$

$$= \frac{P}{EI} \int_0^L x^2 dx$$

$$= \frac{PL^3}{3EI}$$

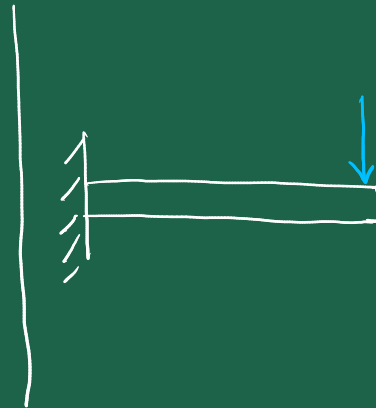
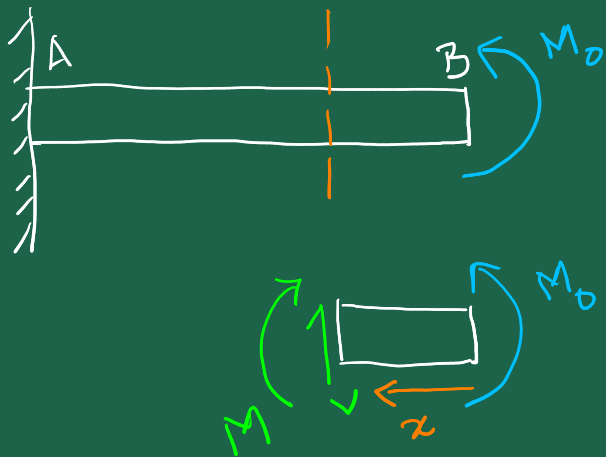
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$$\Delta = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dx$$

Because of symmetry

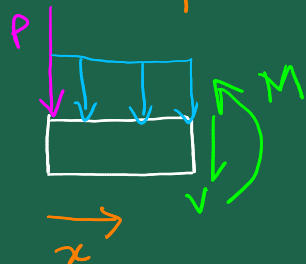
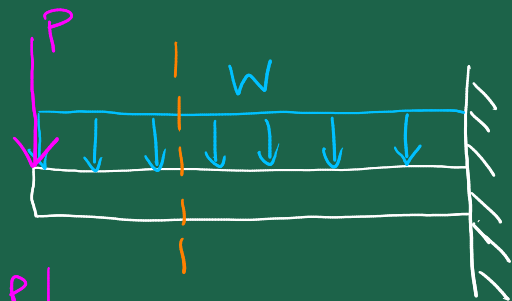
$$\Delta = 2 \int_0^{L/2} \frac{M}{EI} \frac{\partial M}{\partial P} dx$$



$$M = M_0$$

$$\begin{aligned} \theta &= \frac{\partial U}{\partial M_0} = \frac{\partial}{\partial M_0} \int_0^L \frac{M^2}{2EI} dx \\ &= \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_0} dx \\ &= \int_0^L \frac{M_0}{EI} (1) dx = \frac{M_0 L}{EI} \end{aligned}$$

#



$$M + Px + \frac{wx^2}{2} = 0$$

$$\Rightarrow M = -Px - \frac{wx^2}{2}$$

$$\Delta \Big|_{P=0} = \left[\int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dx \right]_{P=0}$$

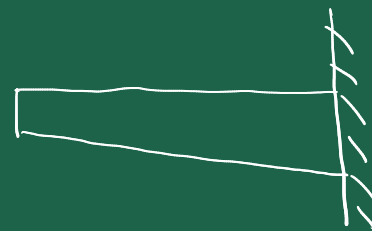
$$= \left[\int_0^L \frac{(-Px - wx^2/2)}{EI} (-x) dx \right]_{P=0}$$

$$= \int_0^L \frac{(-wx^2/2)(-x)}{EI} dx$$

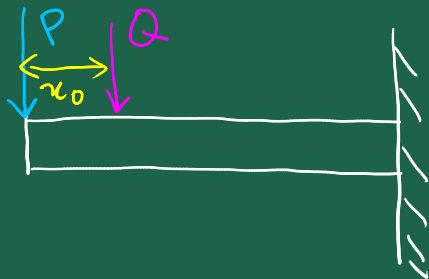
$$= \frac{wL^4}{8EI}$$

$$y = x^2$$

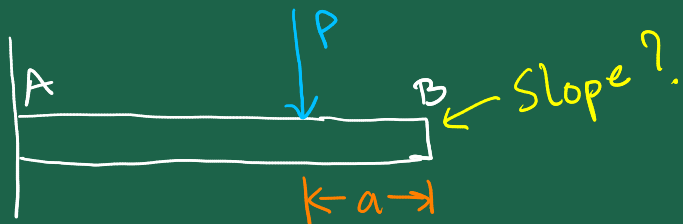
$$\frac{dy}{dx} = 2x$$



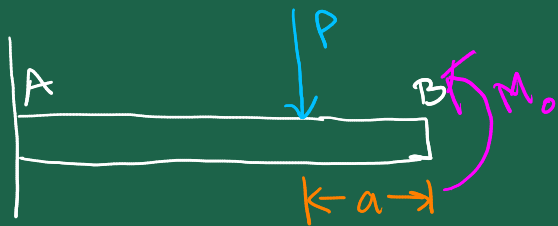
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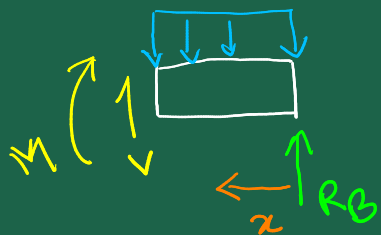
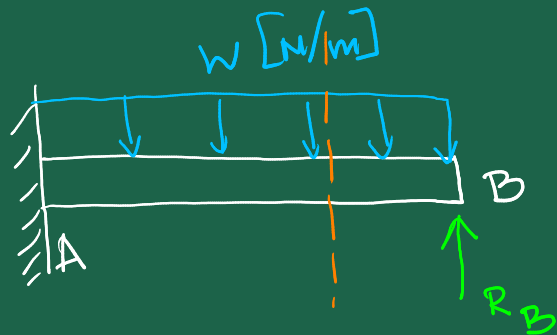
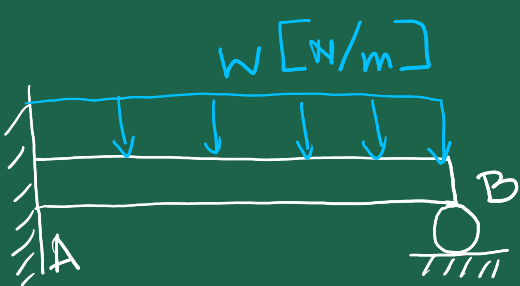


Introduce a dummy or fictitious moment at B.



$$\theta_B|_{M_0=0} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_0} dx$$

#



$$M - R_B x + \frac{wx^2}{2} = 0$$

$$\Rightarrow M = R_B x - \frac{wx^2}{2}$$

0 $\swarrow$ $\searrow$

$$= \int_0^L \frac{M}{EI} \frac{\partial M}{\partial R_B} dx$$

$$= \int_0^L \frac{\left(R_B x - \frac{wx^2}{2} \right)}{EI} (x) dx$$

$$= \frac{R_B}{EI} \int_0^L x^2 dx - \frac{w}{2EI} \int_0^L x^3 dx$$

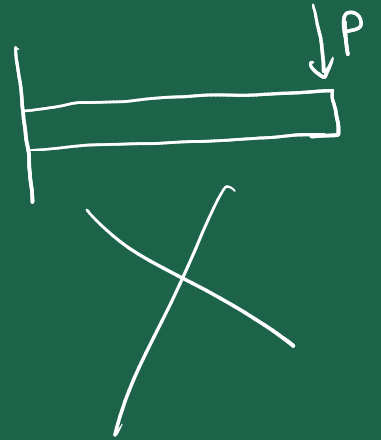
$$= \frac{R_B L^3}{3EI} - \frac{wL^4}{8EI}$$

$$\therefore \frac{R_B L^3}{3EI} = \frac{wL^4}{8EI} \Rightarrow R_B = \frac{3wL}{8}$$

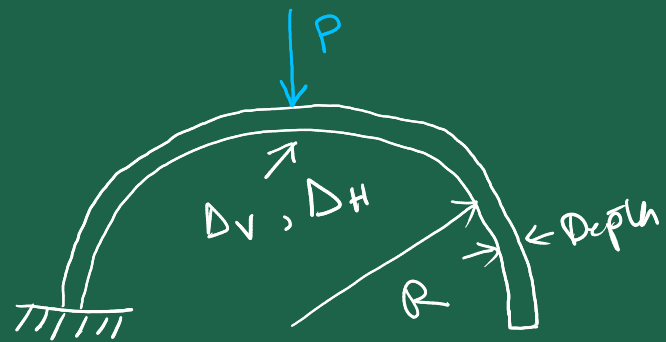
Very Important:



Do NOT use this $\Delta=0$ trick to find reaction forces in statically determinate structures

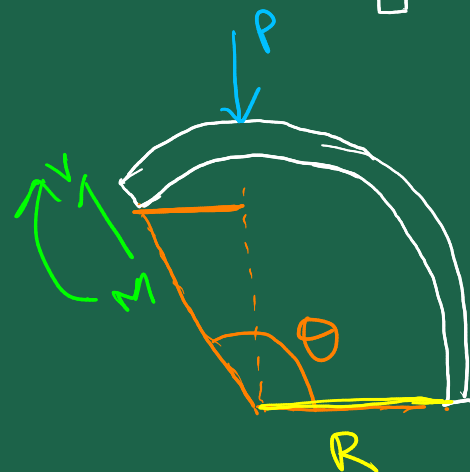
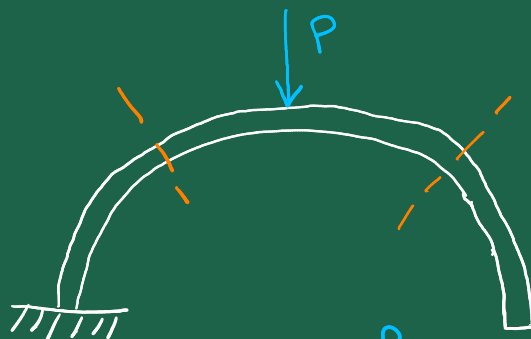


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$R \gg \text{Depth}$
(Thin)

Δ_V :



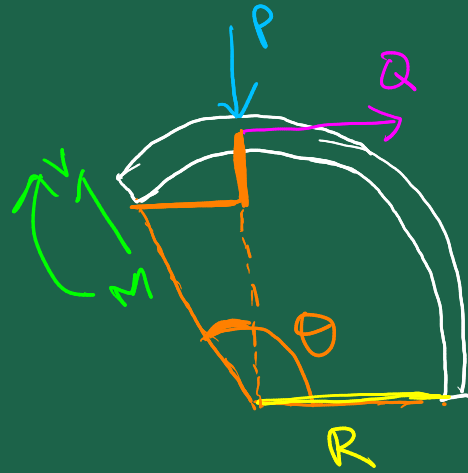
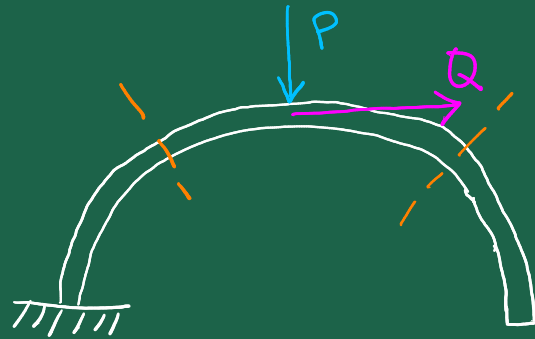
$$M + PR \sin\left(\theta - \frac{\pi}{2}\right) = 0, \quad \frac{\pi}{2} < \theta < \pi$$

$$\Delta_V = \int_0^{\pi} \frac{M}{EI} \frac{\partial M}{\partial P} R d\theta$$

$$\Delta_V = \int_0^{\pi} \frac{M}{EI} \frac{\partial M}{\partial P} R d\theta$$

$$= \int_0^{\pi/2} 0 R d\theta + \int_{\pi/2}^{\pi} \frac{\left\{ -PR \sin(\theta - \pi/2) \right\}}{EI} \left\{ -R \sin(\theta - \pi/2) \right\} R d\theta$$

Δ_H :



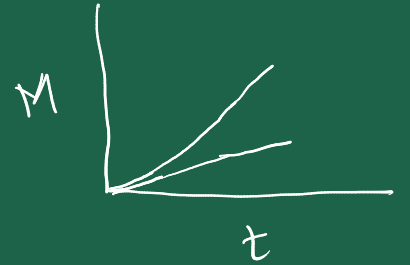
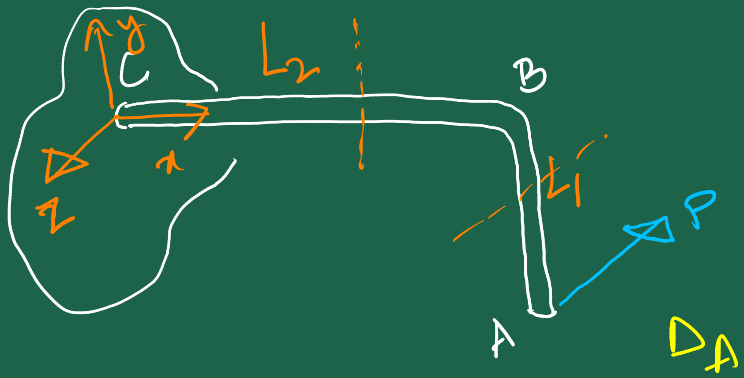
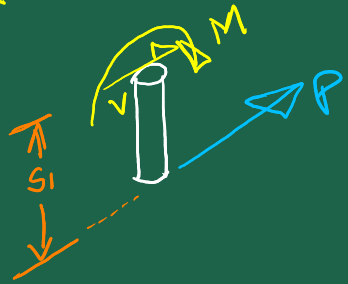
$$M + PR \sin(\theta - \pi/2) + Q \left\{ R - R \cos(\theta - \pi/2) \right\} = 0, \pi/2 < \theta < \pi$$

$$M + PR \sin\left(\theta - \frac{\pi}{2}\right) + Q \left\{ R - R \cos\left(\theta - \frac{\pi}{2}\right) \right\} = 0, \frac{\pi}{2} < \theta < \pi$$

$$\Delta_H \Big|_{Q=0} = \left[\int_0^{\pi} \frac{M}{EI} \frac{\partial M}{\partial Q} R d\theta \right]_{Q=0}$$

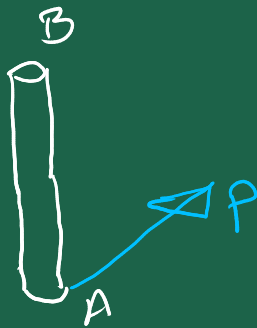
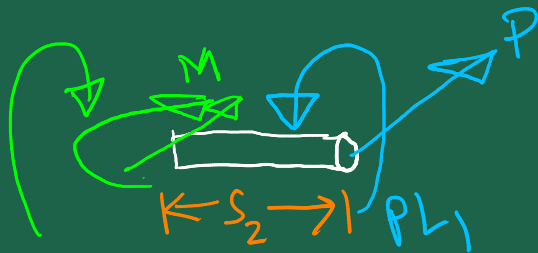
$$= \int_0^{\pi/2} 0 R d\theta + \left[\int_{\pi/2}^{\pi} \frac{M}{EI} \left\{ -R (1 - \cos(\theta - \pi/2)) \right\} R d\theta \right]_{Q=0}$$

#

AB :

$$M - Ps_1 = 0$$

$$\Rightarrow M = Ps_1, \quad 0 < s_1 < L_1$$

BC :

Total strain energy

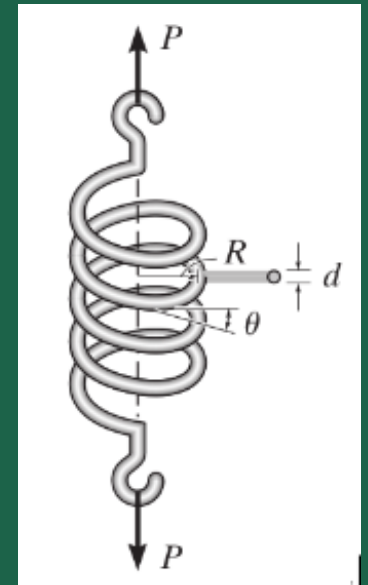
$$U = \underbrace{\int_0^{L_1} \frac{M^2}{2EI} ds_1}_{AB} + \underbrace{\int_0^{L_2} \frac{M^2}{2EI} ds_2 + \int_0^{L_2} \frac{T^2}{2GJ} ds_2}_{BC}$$

$$\Delta_A = \frac{\partial U}{\partial P} = \int_0^{L_1} \frac{M}{EI} \frac{\partial M}{\partial P} ds_1 + \int_0^{L_2} \frac{M}{EI} \frac{\partial M}{\partial P} ds_2 + \int_0^{L_2} \frac{T}{GJ} \frac{\partial T}{\partial P} ds_2$$

$$= \frac{PL_1^3}{3EI} + \frac{PL_2^3}{3EI} + \frac{PL_1^2 L_2}{GJ}$$

2. The coiled spring has n coils and is made from a material having a shear modulus G . Determine the stretch of the spring when it is subject to the load P . Assume that the coils are close to each other so that $\theta \approx 0^\circ$ and the deflection is caused entirely by the torsional stress in the coil.

$$\left[\frac{64nPR^3}{d^4G} \right]$$



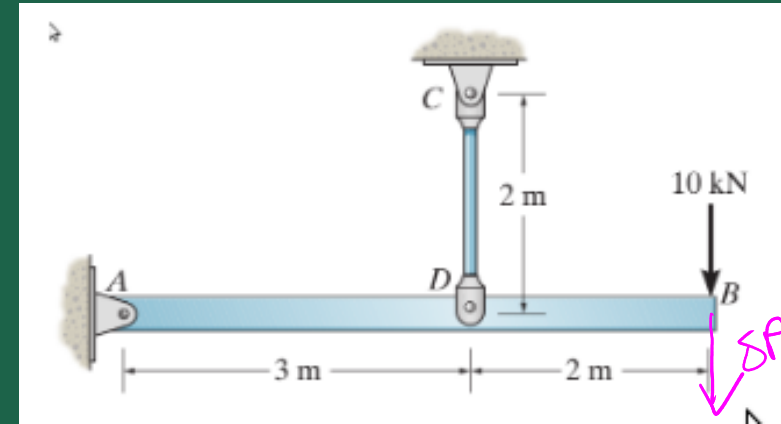
$$T = PR$$

$$U = \int_0^L \frac{T^2}{2GJ} dx = \frac{T^2 L}{2GJ} = \frac{T^2 2\pi R n}{2GJ} = \frac{P^2 R^2 2\pi R n}{2GJ}$$

$$\Delta = \frac{\partial U}{\partial P}$$

$$J = \frac{\pi d^4}{32} \quad \left(\frac{1}{2} \pi d^4 \right)$$

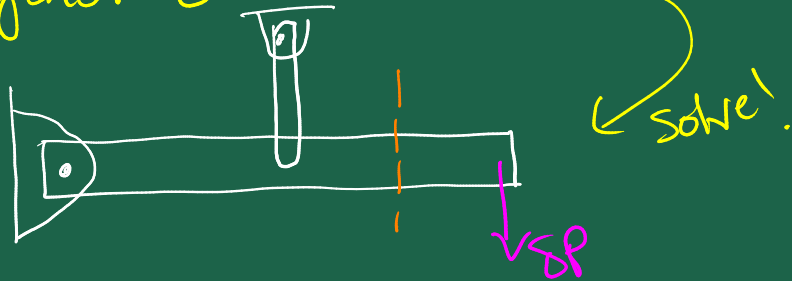
6. Beam AB has a square cross section of 100 mm by 100 mm. Bar CD has a diameter of 10 mm. If both members are made of steel ($E = 200 \text{ GPa}$), determine the vertical displacement of point B and the slope at A. [43.5 mm, 0.00530 rad]



PVN

Remove the ext. loading and apply δP

$\delta P \rightarrow$ generate some internal forces



Bring back the external loading and consider all the internal deformations

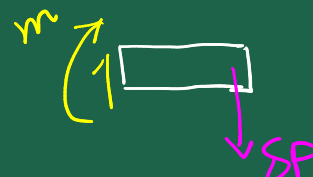
$$\delta P \times \Delta = \sum \delta F \times u$$

$$\frac{FL}{AE}$$

$$1 \times \Delta_B = \underbrace{\Delta_{AB}}_{\substack{B \\ D}} + \underbrace{\Delta_{CD}}_{\substack{A \\ D}} + \delta F_D \left(\frac{FL}{AE} \right)$$

$$\int_B^D = \int_B^D m d\theta = \int_0^2 m \frac{M dx}{EI}$$

$$= \int_0^2 (-x) \frac{(-10x)}{EI} dx$$



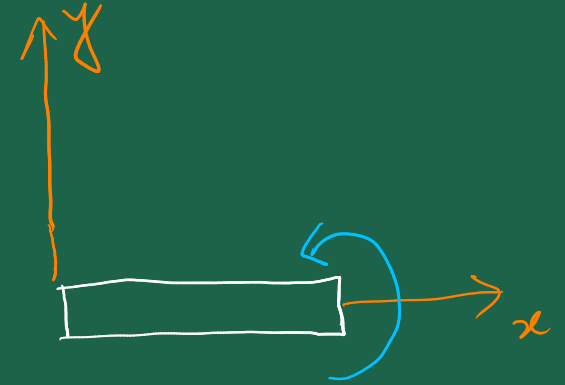
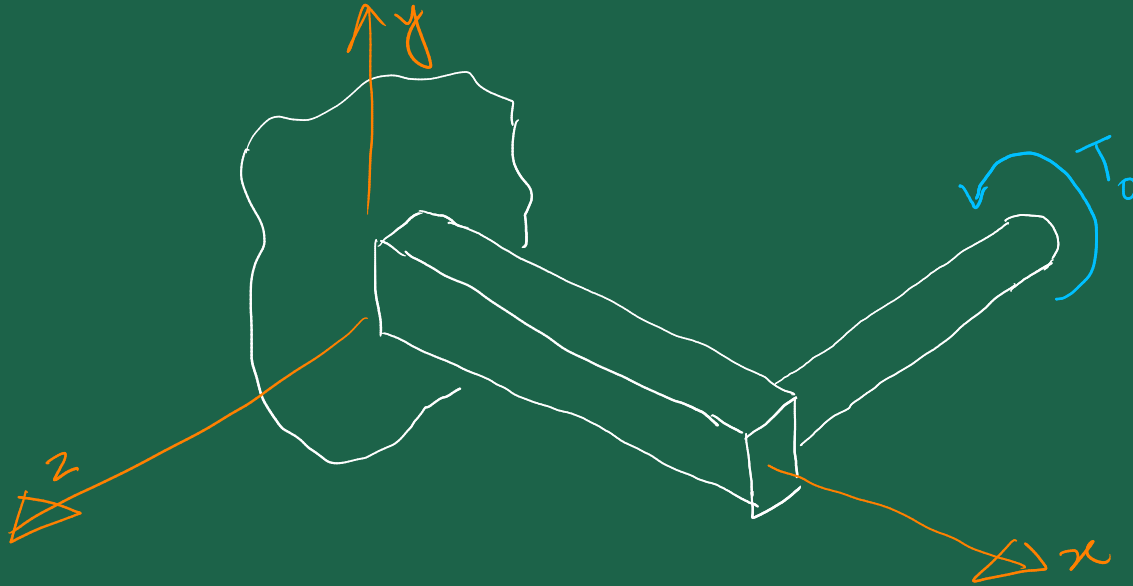
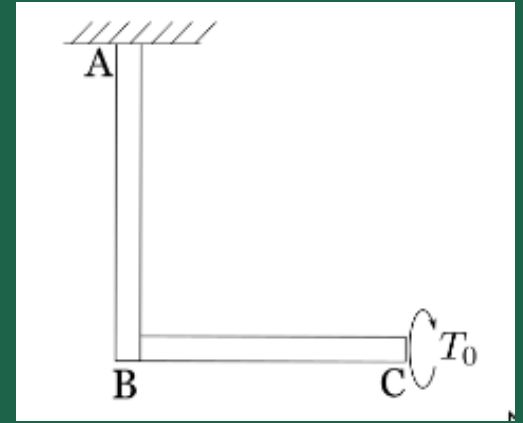
$$m = -\delta P x$$

$$= -x$$

Complete the rest of the steps yourself!

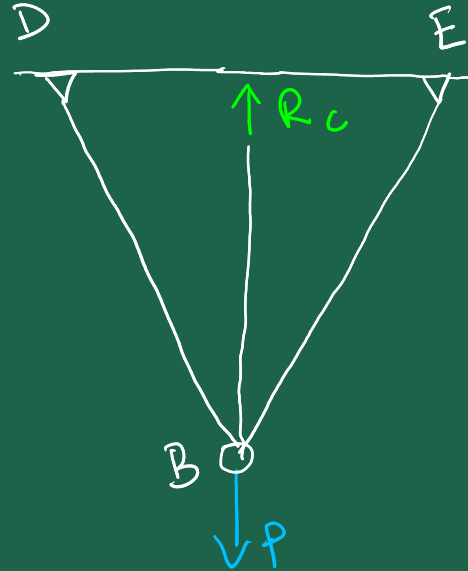
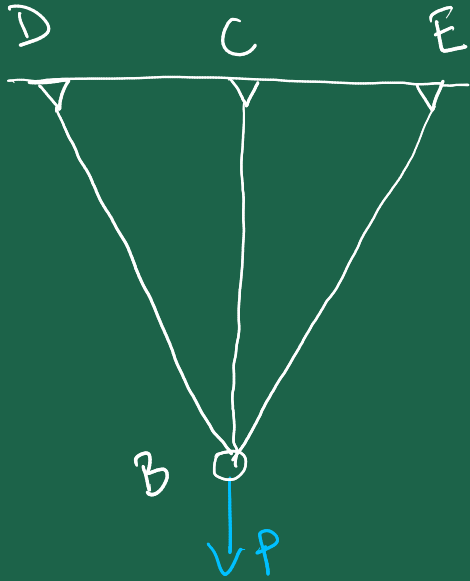
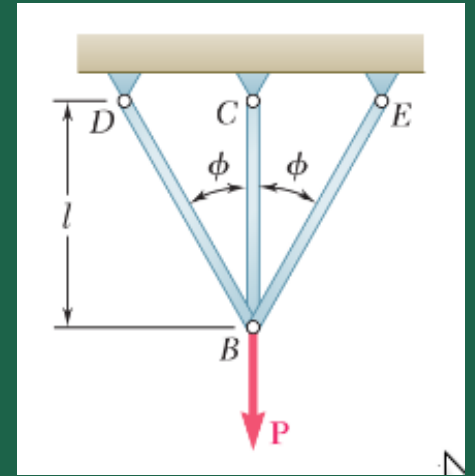
Compare the above solⁿ based on PVW with the solution based on Castigliano's theorem

10. A shaft BC (length: 1.2 m) of circular cross-section (diameter 60 mm) is welded to a beam AB (length 1.5 m) of rectangular cross-section (70 mm $\times$ 50 mm). A torque $T_0 = 2.50 \text{ kN}\cdot\text{m}$ applied at C as shown. Determine the rotation of the end C. Both the shaft and the beam are made of steel ($E = 200 \text{ GPa}$; $G = 77.5 \text{ GPa}$). [0.0523 rad]



11. Three members of the same material and same cross-sectional area are used to support the load P . Determine the force in the member BC.

$$\left[\frac{P}{1 + 2 \cos^3 \phi} \right]$$



$$2F_0 \cos \phi + R_C = P$$

$$\Rightarrow F_0 = \frac{P - R_C}{2 \cos \phi}$$

$$L_{BD} = L_{BE} = \frac{l}{\cos \phi}$$

$$\Delta_C = \frac{\partial U}{\partial R_C} = \frac{\partial U_{BD}}{\partial R_C} + \frac{\partial U_{BE}}{\partial R_C} + \frac{\partial U_{BC}}{\partial R_C}$$

$$= \frac{\partial}{\partial R_C} \left(\frac{F_0^2 L_{BD}}{2AE} \right) + \frac{\partial}{\partial R_C} \left(\frac{F_0^2 L_{BE}}{2AE} \right) + \frac{\partial}{\partial R_C} \left(\frac{R_C^2 l}{2AE} \right)$$

$$= 2 \frac{\partial}{\partial R_C} \left(\frac{\vec{F}_0^T L_{BD}}{2AE} \right) + \frac{\partial}{\partial R_C} \left(\frac{R_C^T l}{2AE} \right)$$

$$F_0 = \frac{P - R_C}{2 \cos \phi}$$

$$= 2 \frac{2F_0}{2AE} \frac{l}{\cos \phi} \frac{\partial F_0}{\partial R_C} + \frac{2R_C l}{2AE}$$

$$\Rightarrow \frac{\partial F_0}{\partial R_C} = \frac{-1}{2 \cos \phi}$$

$$= \frac{2F_0}{AE} \frac{l}{\cos \phi} \left(\frac{-1}{2 \cos \phi} \right) + \frac{R_C l}{AE}$$

But we know that $\Delta_C = 0$

$$\therefore \frac{2F_0 l}{AE \cos \phi} \left(\frac{+1}{2 \cos \phi} \right) = \frac{R_C l}{AE}$$

$$\Rightarrow \frac{P - R_C}{2 \cos^3 \phi} = R_C \Rightarrow R_C = \frac{P}{1 + 2 \cos^3 \phi}$$