Digital Image: A function over space

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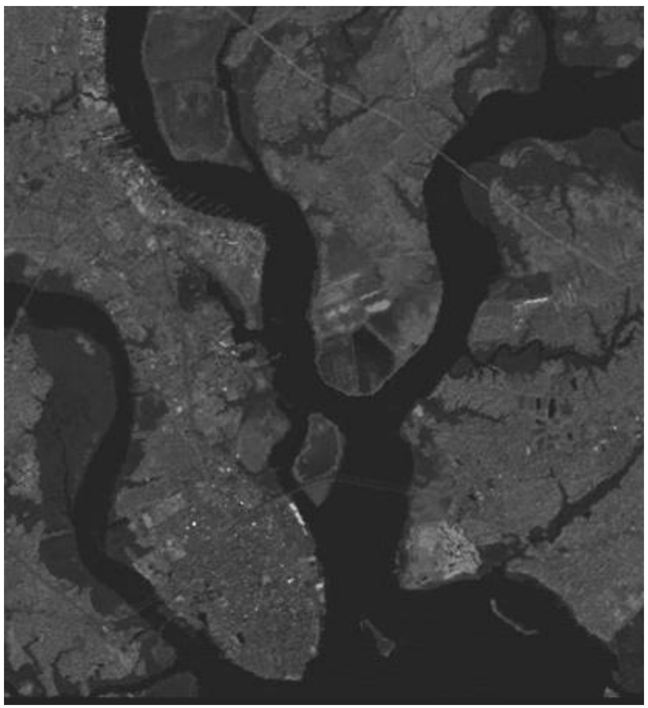
Image as a function over space

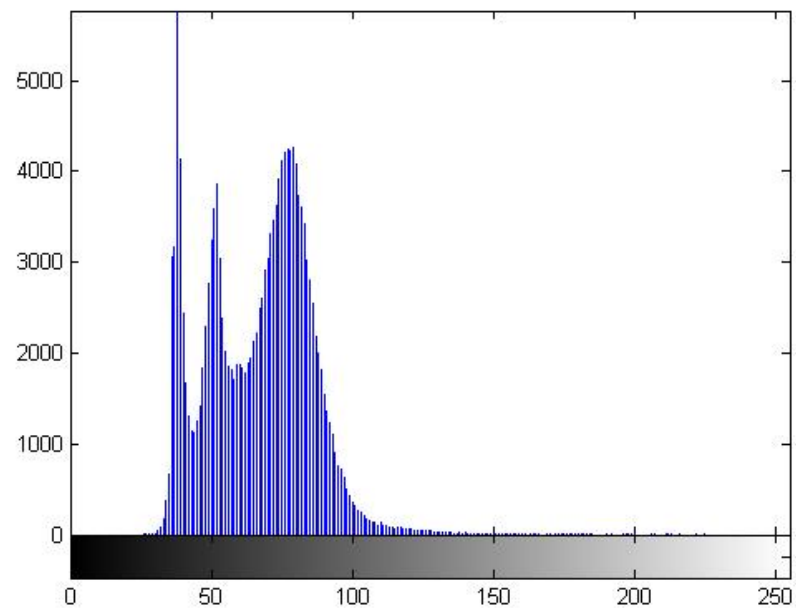
- Continuous function:
  - f: $R^2 \rightarrow R$ (or C)
  - f: $R^n \rightarrow R$ (or C)
- Discrete function:
  - f: $Z^2 \rightarrow R$ (or Z) (2-D image)
  - f: $Z^n \rightarrow R$ (or Z) (n-D)
- Semantics of f(x,y)
  - Brightness value
  - Depth value
  - Absorption coefficients

- Apply various processing
  - f(.) as a random variable.
  - The discretized grid as a random field.
  - Resizing (upsampling and downsampling)
  - Pixel mapping.
  - Compute gradients and higher order derivatives.
  - Functional decomposition.
  - Correlation, convolution and filtering

Image histogram



Frequency



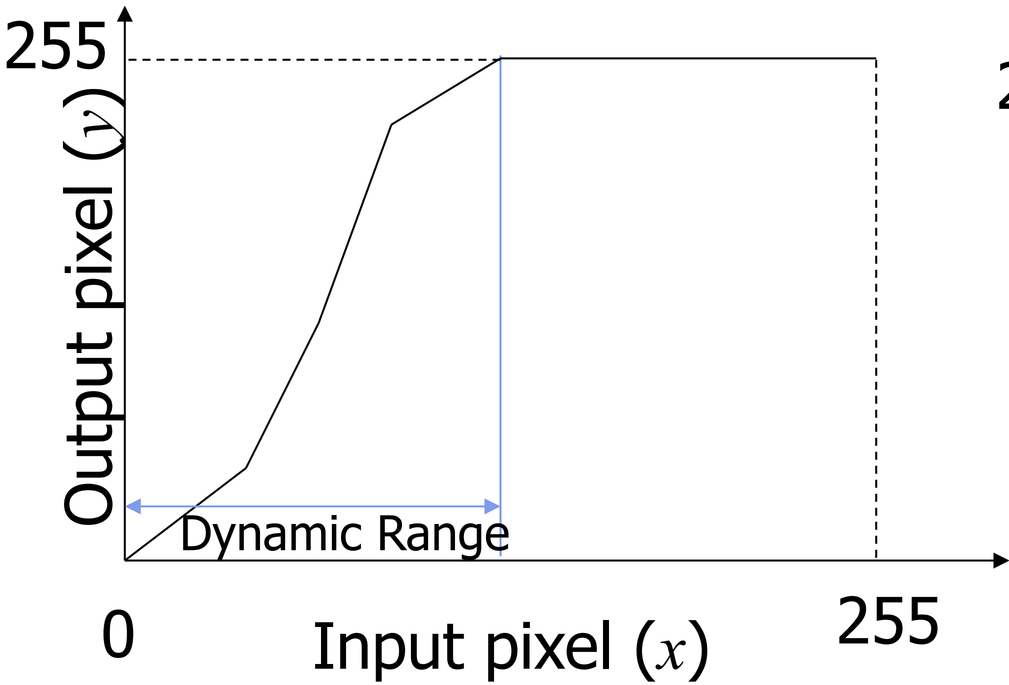
Brightness (x)

Normalized Histogram → p(x)

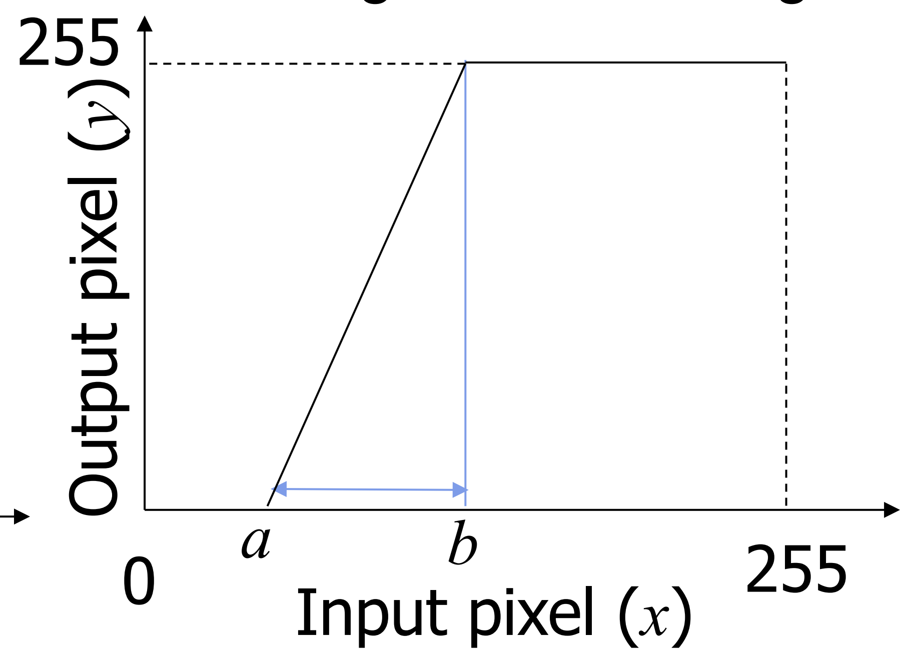
Contrast Enhancement: Pixel mapping

Function is monotonically increasing.

If $x_1 > x_2$, $y_1 \geq y_2$



Histogram Stretching

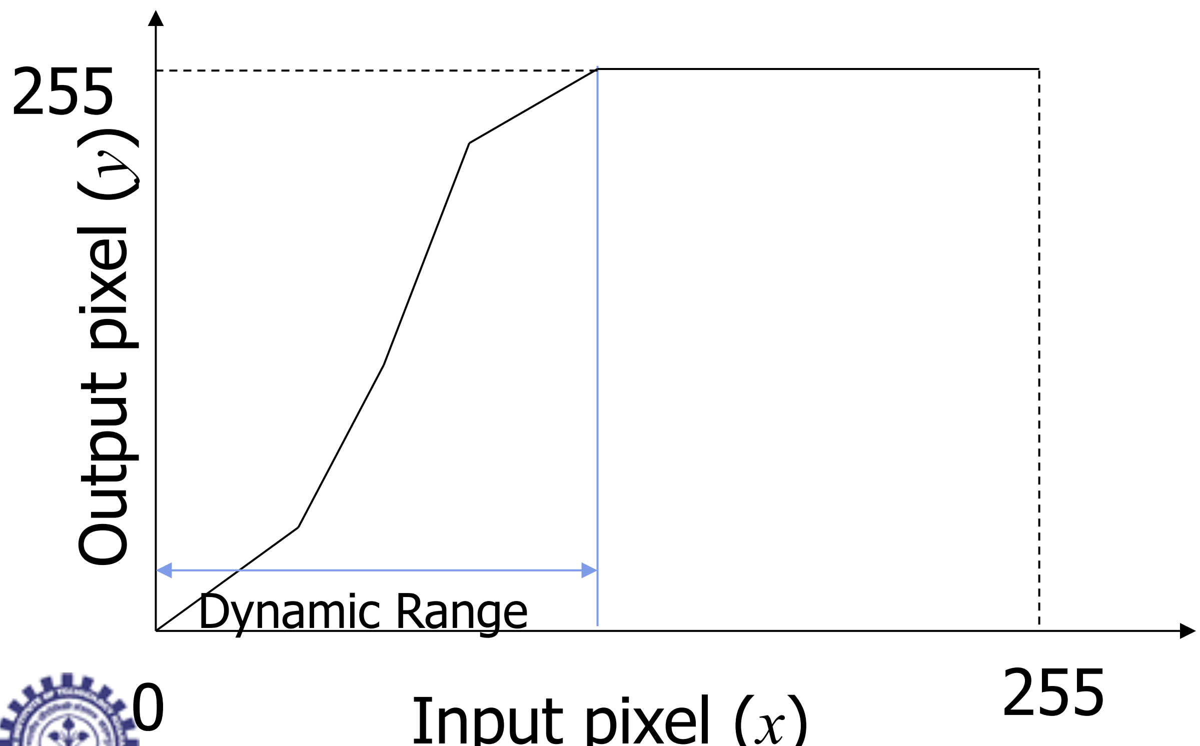


$$y = \frac{255}{b-a}(x-a), a \leq x \leq b$$

Histogram Equalization

A fundamental theorem of probability density function

If $y=T(x),\ p(y)=p(x).|dx/dy|$



$$y = 255.\sum_{t=0}^{x} p(t)$$

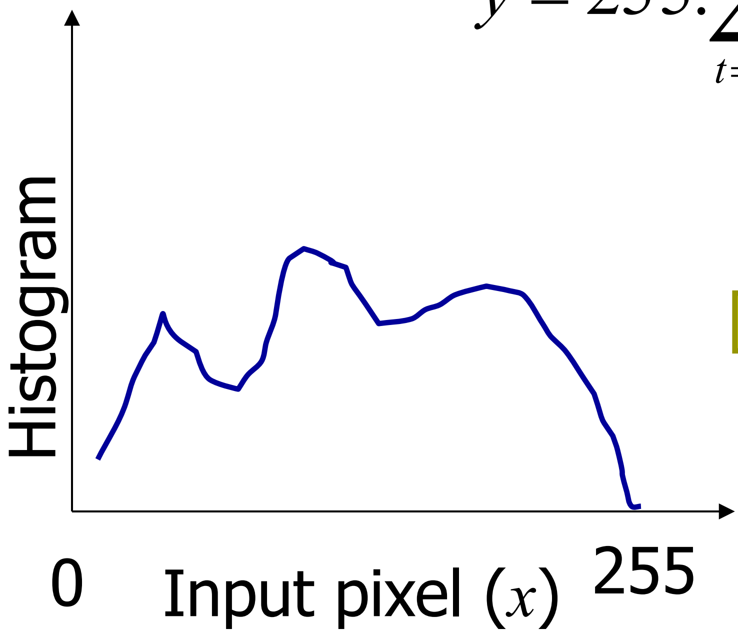
$$dy/dx = 255.\ p(x)$$

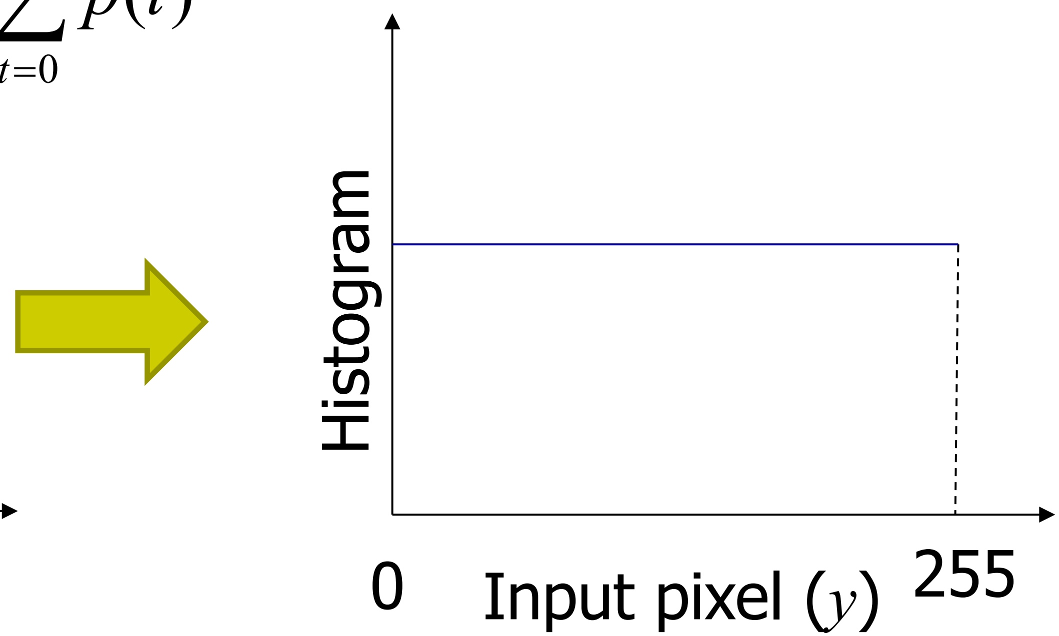
$$p(y) = 1/255$$

Uniform distribution

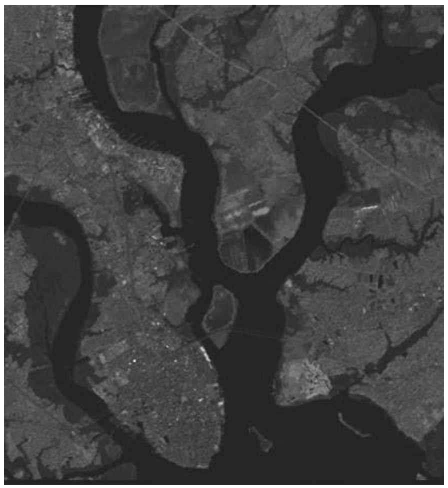
Histogram equalization

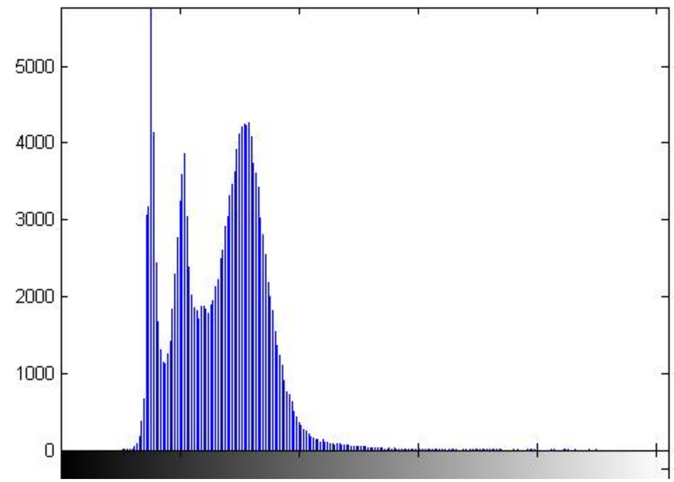
$$y = 255.\sum_{t=0}^{x} p(t)$$

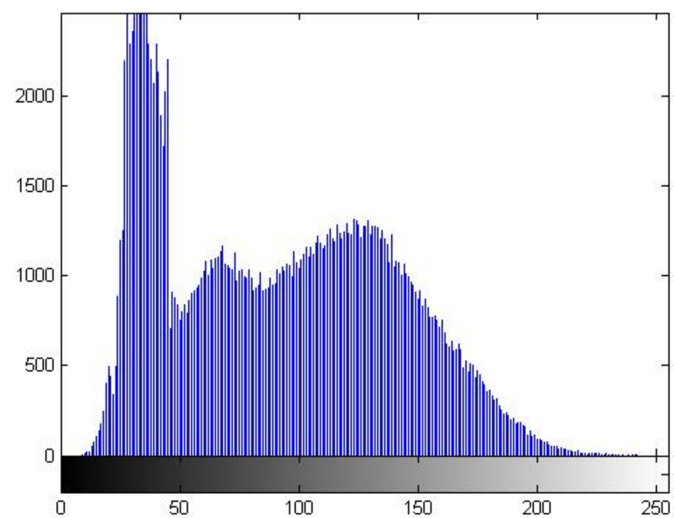




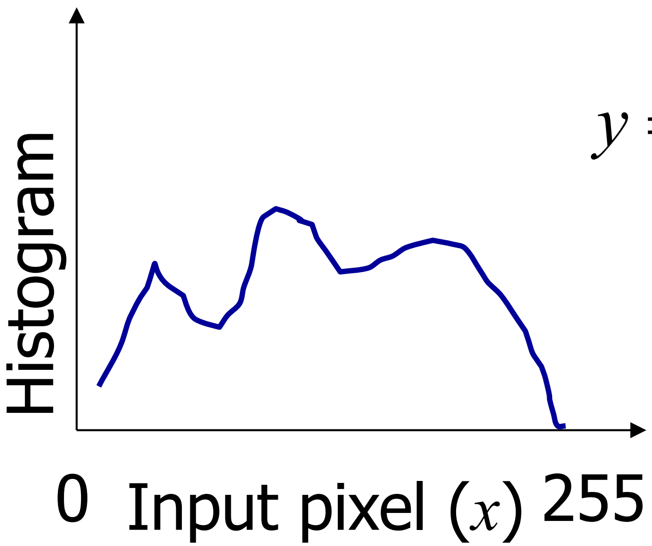
Histogram Equalization



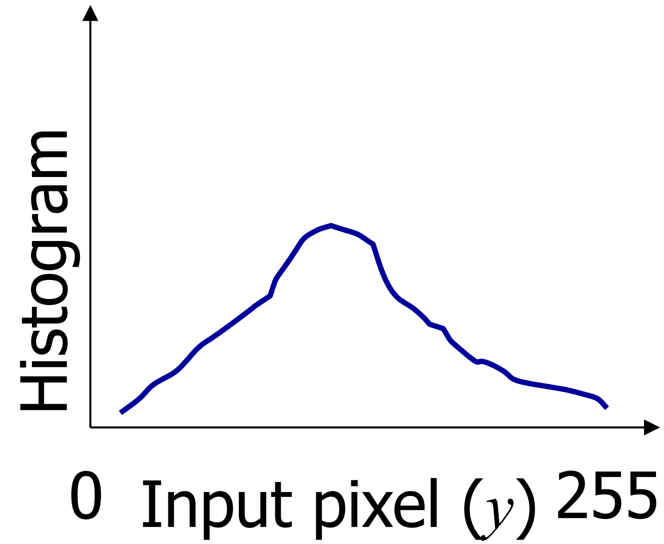




General histogram equalization



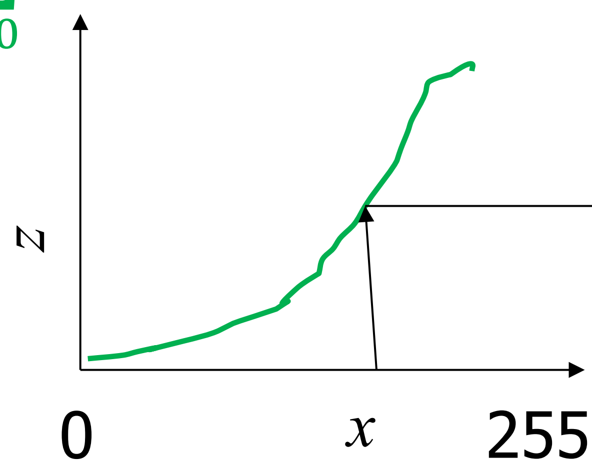
$$y = 255.\sum_{t=0}^{x} p(t)$$

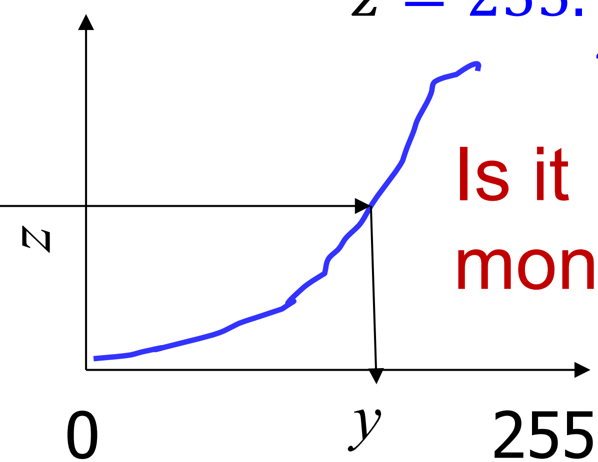


$$z = 255.\sum_{t=0}^{x} p_x(t)$$

x → y , if z(x)=z(y)

$$z = 255.\sum_{t=0}^{y} p_y(t)$$





Is it monotonic?

Other pixel mapping functions

log function: $y = c.\log(1 + x)$

Power law (Gamma) transformation:

n-th root: $y = c.\sqrt[n]{x}$

n-th power: $y = c.x^n$

$y = c.x^{\gamma}$

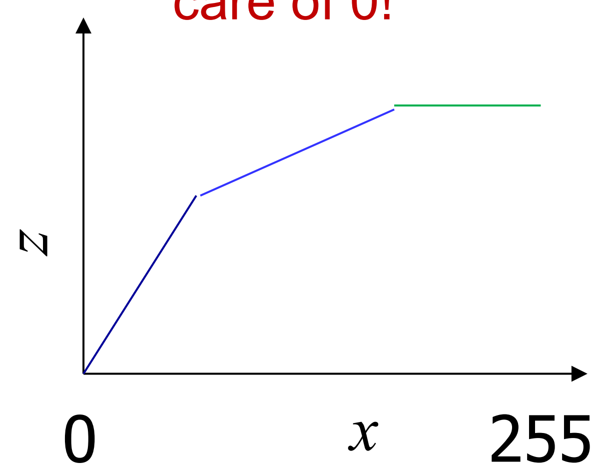
$y = c.(x + \varepsilon)^{\gamma}$

Any real positive number

Offset to take care of 0!

Inverse log function: $y = c.(e^x - 1)$

Piece wise linear function:

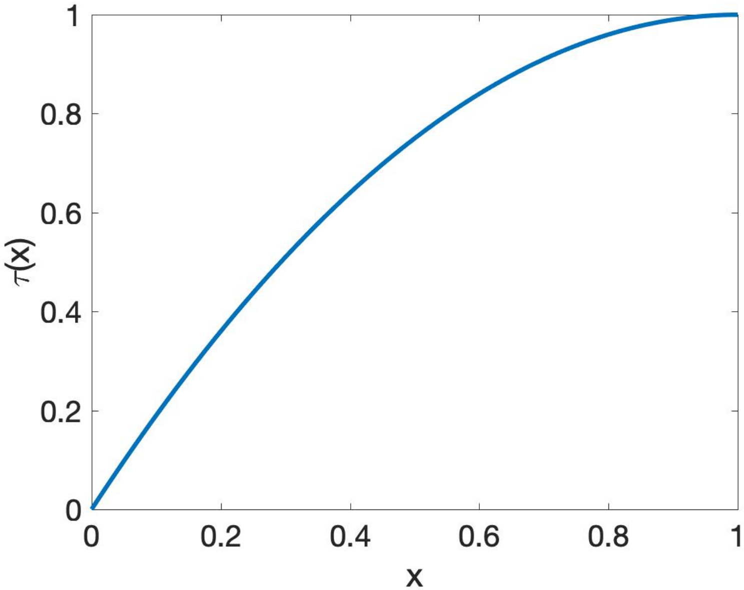


Negative function:

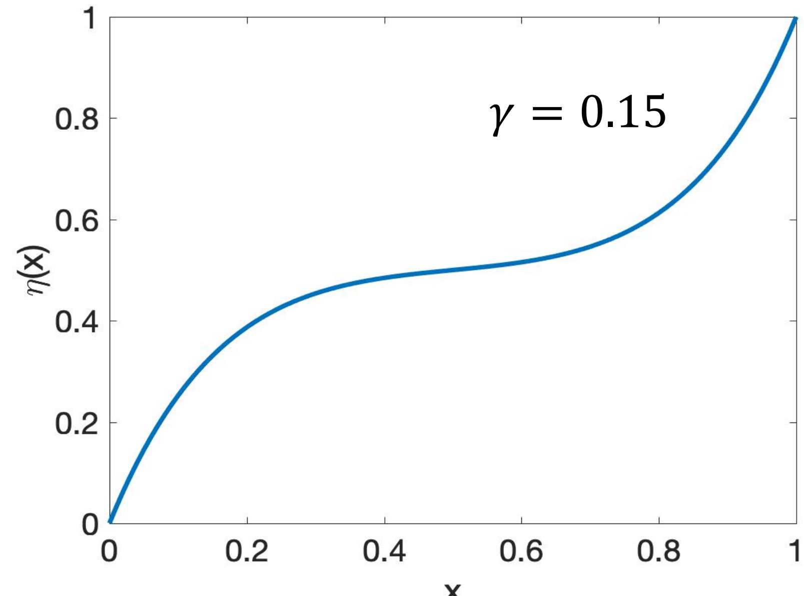
$y = 255 - x$

Which functions are suitable for contrast enhancement?

Two interesting functions for contrast stretching



$$\tau(x) = x(2 - x)$$



$$\eta(x) = \frac{\left(x^{\frac{1}{\gamma}} + \left(1 - x^{\frac{1}{\gamma}}\right)\right)}{2},$$

$$0 \le x \le 1$$

Adaptive scaling of pixel value

- Compute global mean (m_g) and global std. dev. (s_g). of an image.
- At a pixel compute local mean (m_l) and local s.d (s_l).
- If $m_l < k_m\, m_g$ and $s_l < k_s\, s_g$, and also ($s_l > k_{min}.\, s_g$) (for retaining constant regions), scale the value by a constant factor (say E), else no scaling.
  - To enhance darker region $0 < k_m < 1$
  - For taking care of local contrast $k_s < 1$ if it is darker, else >1 if it is a brighter part.
  - $k_{min} < k_s$
  - E should be > 1 or < 1 to perform increase or decrease the intensity level compared to the original image

Adaptive scaling of intensity value

- Compute global mean (m_g) and global (s_g). of an image.
- At a pixel compute local mean (m_l) and local s.d (s_l).
- If the local mean and local s.d. are less than certain factors (k_m and k_s) of global mean and s.d. and also if the local s.d. is greater than a minimum value ($k_{min}. s_g$) (for retaining constant regions), scale the value by a constant factor.
  - To enhance darker region k_m and k_s should be > 1.

Gradient Operations

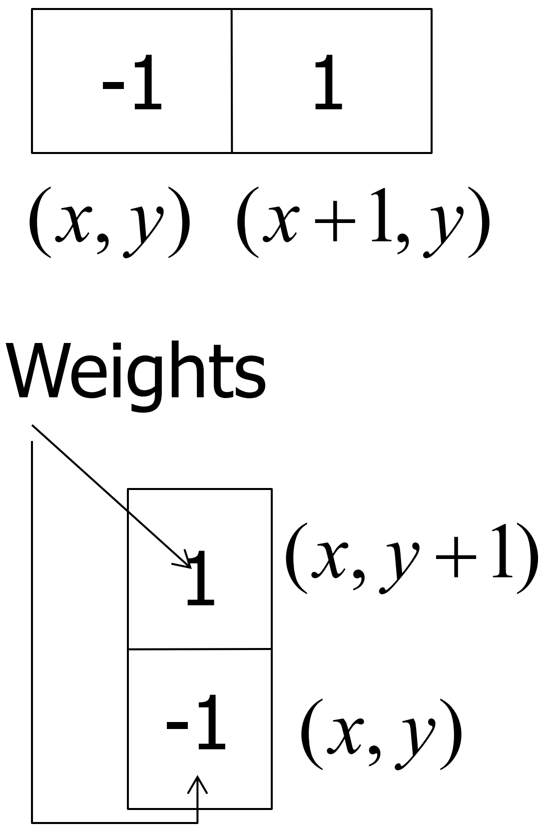
Consider the image as a 2D function: $f(x, y)$

$$\frac{\partial f(x,y)}{\partial x} = f(x+1, y) - f(x, y)$$

$$\frac{\partial f(x,y)}{\partial y} = f(x, y+1) - f(x, y)$$

$$\nabla f(x,y) = \frac{\partial f(x,y)}{\partial x}\hat{i} + \frac{\partial f(x,y)}{\partial y}\hat{j}$$

Computation with mask



1. Scan the image top to bottom and left to right.
2. At every point (x,y) place the mask and compute the weighted sum.

$$g(x,y)=(1).f(x,y+1)+(-1).f(x,y)$$

3. Write the value $g(x,y)$ at (x,y) pixel position of the processed image.

Robust gradient computation

Averaging neighboring gradient values

-1	-1 1	1
-1	-1 1	1
-1	-1 1	1

$\Rightarrow$

-1	0	1
-1	0	1
-1	0	1

1	1	1
0	0	0
-1	-1	-1

Prewitt operator

(6 times of the gradient value in any direction)

Robust gradient computation

Weighted average of neighboring gradient values

1x	-1	-1 1	1
2x	-1	-1 1	1
1x	-1	-1 1	1

-1	0	1
-2	0	2
-1	0	1

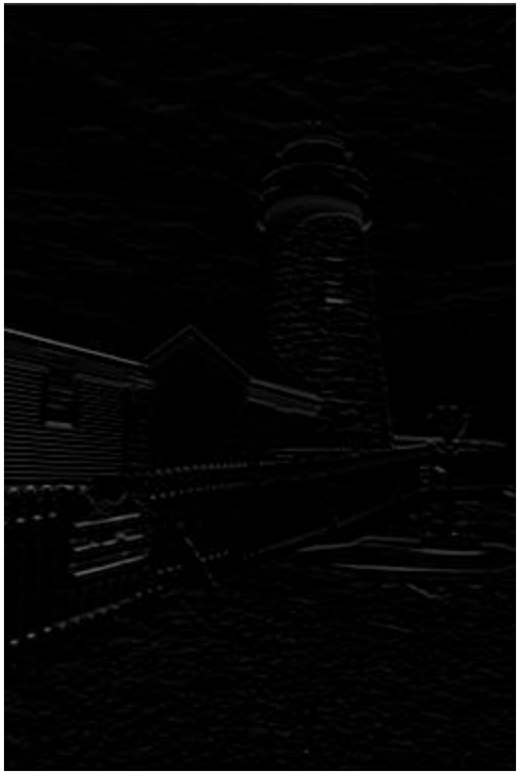
1	2	1
0	0	0
-1	-2	-1

Sobel operator

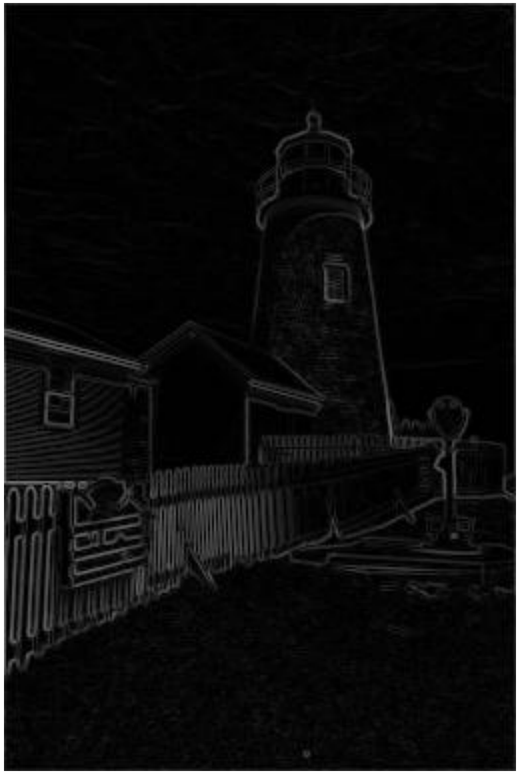
(8 times of the gradient value in any direction)

Results of gradient operations

Vertical



Horizontal



Resultant

Higher order gradients!

$$\frac{\partial f(x,y)}{\partial x} = f(x+1,y) - f(x,y)$$

1	-2	1
(x-1,y)	(x,y)	(x+1,y)

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x-1,y)}{\partial x}$$

$$\approx f(x+1,y) - f(x,y) - (f(x,y) - f(x-1,y))$$

$$= f(x+1,y) - 2f(x,y) + f(x-1,y)$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

0	1	0
1	-4	1
0	1	0

OR

1	0	1
0	-4	0
1	0	1

Laplacian Operator

Weighted combination!

Laplacian Operator

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

$$1 \times \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 4 \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{5} \times \begin{bmatrix} 4 & 1 & 4 \\ 1 & -20 & 1 \\ 4 & 1 & 4 \end{bmatrix}$$

Image sharpening

How do you sharpen a color image?

- Add a factor proportional to the gradient values to the pixel values.
  - Sharpens edges as gradients are high there.

$$g(x,y) = f(x,y) + c.||\nabla f(x,y)||$$

$c>0$

- To make it insensitive to direction Laplacian operator may be used.
  - The center value is subtracted. Hence the contribution to be subtracted.

$$g(x,y) = f(x,y) - c.||\nabla^2 f(x,y)|| \quad c>0$$

More on computation with mask: Correlation Filter

w_1	w_2	w_3
w_4	w_c	w_5
w_6	w_7	w_8

Cross Correlation:

$$g(x,y) = w(x,y) \bullet f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x+s, y+t)$$

$$\boxed{\begin{aligned} g(x,y) = {} & w_1 f(x-1,y+1) + w_2 f(x,y+1) + w_3 f(x+1,y+1) + w_4 f(x-1,y) + \\ & w_c f(x,y) + w_5 f(x+1,y) + w_6 f(x-1,y-1) + w_7 f(x,y+1) + w_8 f(x+1,y+1) \end{aligned}}$$

$$f(x,y) \longrightarrow \boxed{w(x,y)} \longrightarrow g(x,y)$$

Template Matching:

Mask→ Template weights ($w(x,y)$)

Convolution

Linear Shift Invariant System

$$\delta(x,y) \longrightarrow \boxed{\text{LSI}} \longrightarrow w(x,y)$$

w_1	w_2	w_3
w_4	w_c	w_5
w_6	w_7	w_8

$$g(x,y) = w(x,y) * f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x-s, y-t)$$

Same as correlation except that the mask is flipped both horizontally and vertically.

$$\boxed{\begin{aligned} g(x,y) &= w_8 f(x-1,y+1) + w_7 f(x,y+1) + w_6 f(x+1,y+1) + w_5\, f(x-1,y) + \\ & w_c f(x,y) + w_4 f(x+1,y) + w_3 f(x-1,y-1) + w_2 f(x,y+1) + w_1 f(x+1,y+1) \end{aligned}}$$

Filtering: $f(x,y) \longrightarrow \boxed{w(x,y)} \longrightarrow g(x,y)$

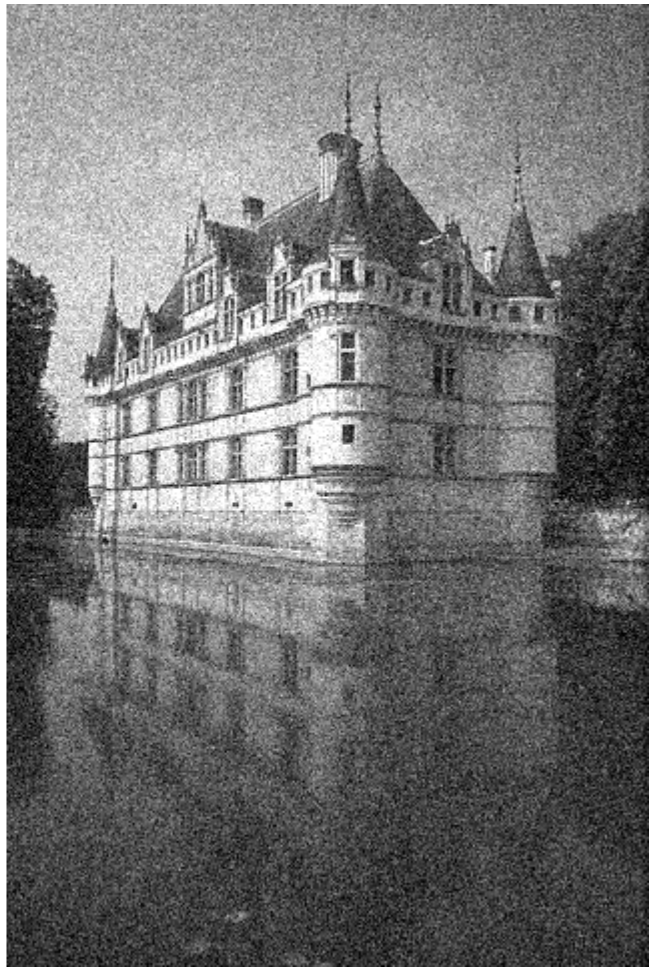
If w(x,y) is symmetric, that is w(x,y)=w(-x,-y), then convolution is equivalent to correlation!

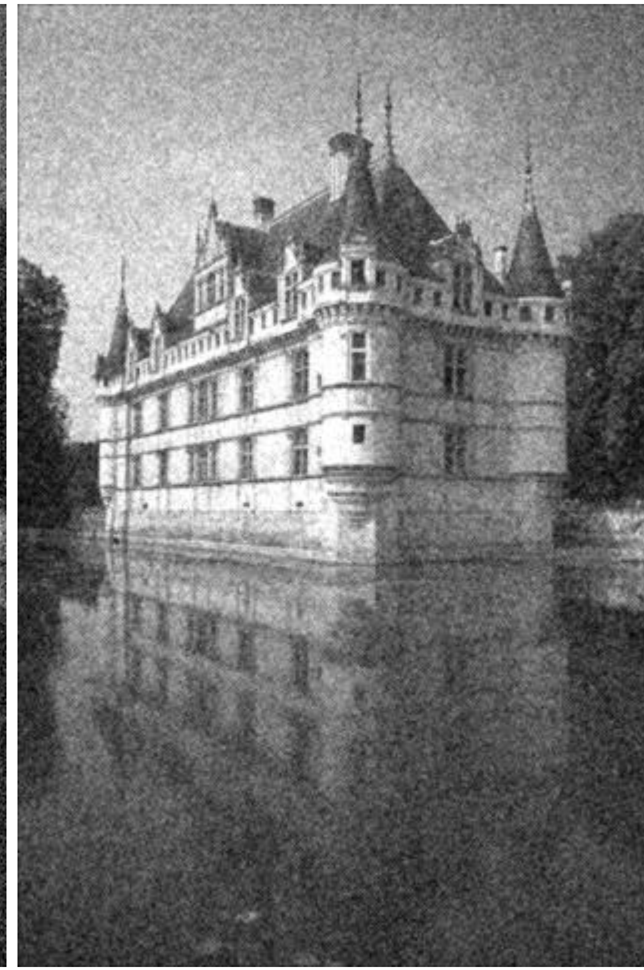
Additive Noise Filtering

Weighted Mean filter: sum of the weights= 1, and all positive.

c	b	c
b	a	b
c	b	c

$a=0.5,\ b=0.3/4,$ and, $c=0.2/4$





Gaussian Smoothing

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{-((x-x_c)^2+(y-y_c)^2)}{2\sigma^2}}$$

$$g(x,y) = f(x,y) * G(x,y)$$

σ=2

Mask size: 9x9

Is it a weighted mean filter?

How to make it?

Nonlinear filters

Median Filter

$g(x,y) =$ the median value among the neighbors including (x,y).

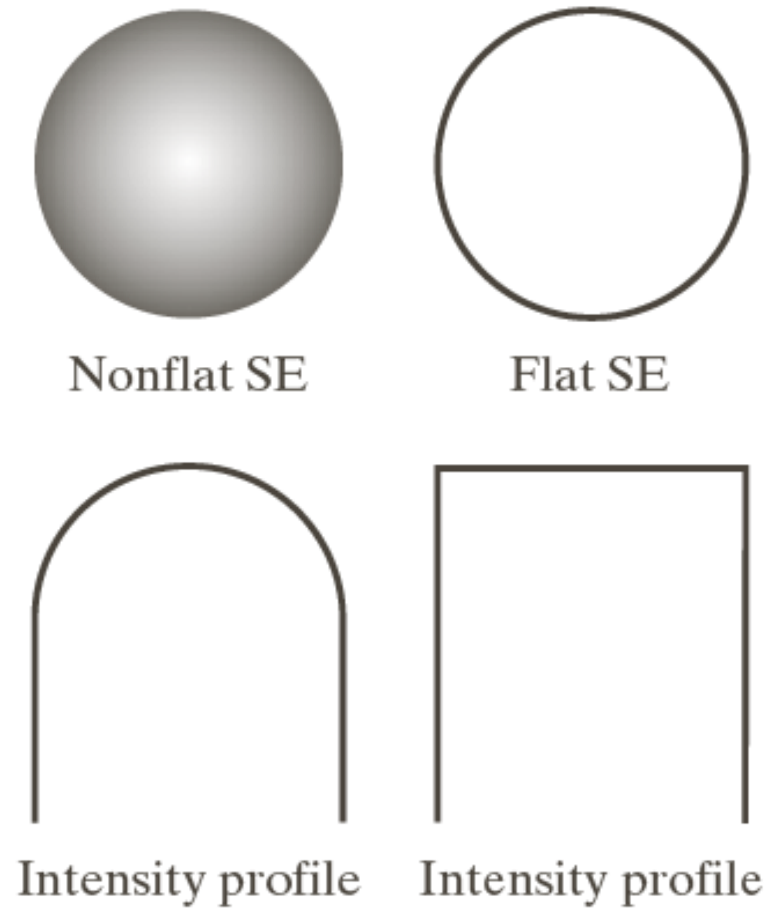
Weighted median filter?

Repeat the neighbors by times of its integral weight and take the median.

More general form:
Order statistics filter
Linear combination of ranked order neighbors.

Gray-Scale Morphology

- f(x,y): Gray-Scale Image
- b(x,y): Structuring Element
  - Non-Flat
  - Flat
- Coordinates
  - Z^2
- Intensity
  - R or Z



Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Gray-Scale Morphology

- Typical SE
  - Symmetrical
  - Flat
  - Unit Height
  - Origin @ Center
- Reflection
  - b^(x,y) = b(-x,-y)

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Gray-Scale Dilation

$$(f \oplus b)(s,t) = \max_{(x,y)}\{f(s-x,t-y)+b(x,y) \mid (s-x,t-y)\in D_f;(x,y)\in D_b\}$$

- $f(x,y)$: Input image
- $b(x,y)$: SE
- Similar to 2-D convolutions
- General effect:
  - If all elements of SE +ve, output gets brighter.
  - Dark details get either reduced or eliminated

Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Gray-Scale Dilation

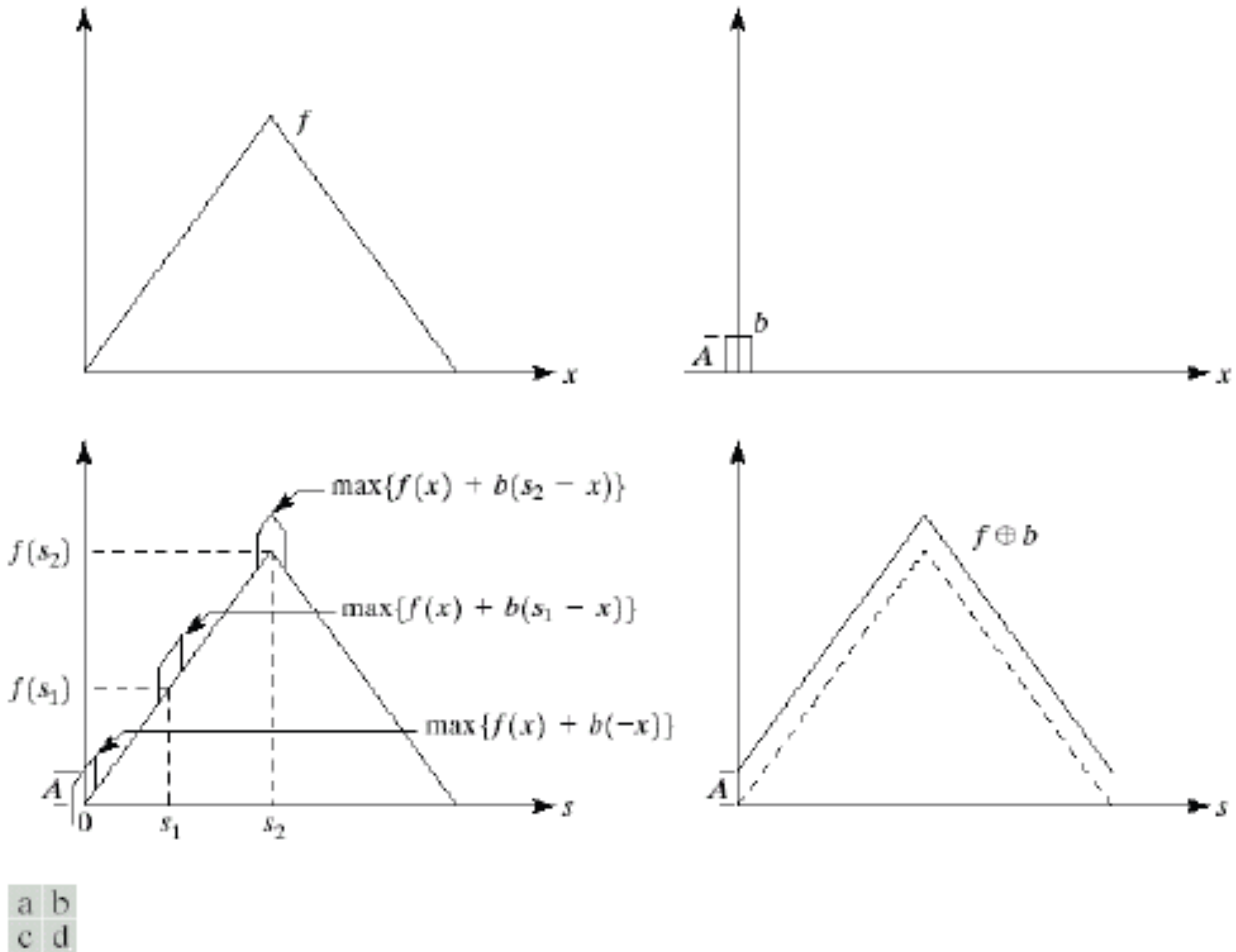


FIGURE 9.27 (a) A simple function. (b) Structuring element of height A. (c) Result of dilation for various positions of sliding b past f. (d) Complete result of dilation (shown solid).

Gray-Scale Erosion

$$(f \ominus b)(s,t) = \min_{(x,y)}\left\{f(s+x,t+y) - b(x,y) \mid (s+x,t+y) \in D_f; (x,y) \in D_b\right\}$$

- *f(x,y)* : Input image
- *b(x,y)*: SE
- Similar to 2-D correlation or template matching
- General effect:
  - If all elements of SE +ve, output gets darker.
  - Brighter details get reduced and smaller areas than SE gets eliminated

Erosion / Dilation: Flat SE

$$[f \ominus b](x, y) = \min_{(s,t)\in b}\{f(x+s, y+t)\}$$

$$[f \oplus b](x, y) = \max_{(s,t)\in b}\{f(x-s, y-t)\}$$

Gray-Scale Dilation & Erosion

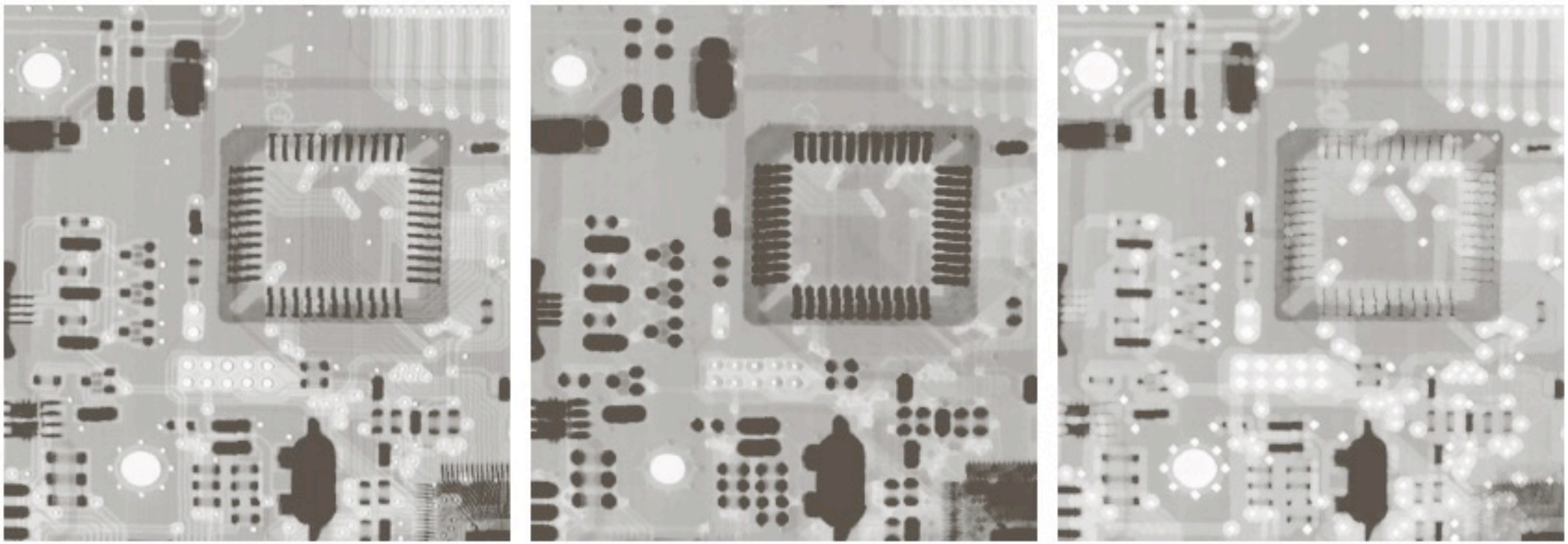
- Duals with respect to function complementation and reflection.

$$(f \ominus b)^c(s,t) = (f^c \oplus \hat{b})(s,t)$$

Where

$$f^c(s,t) = -f(s,t) \qquad \hat{b}(s,t) = b(-s,-t)$$

Gray-Scale Dilation & Erosion



X-Ray image

Erosion using flat disk SE with a radius of 2 pixels

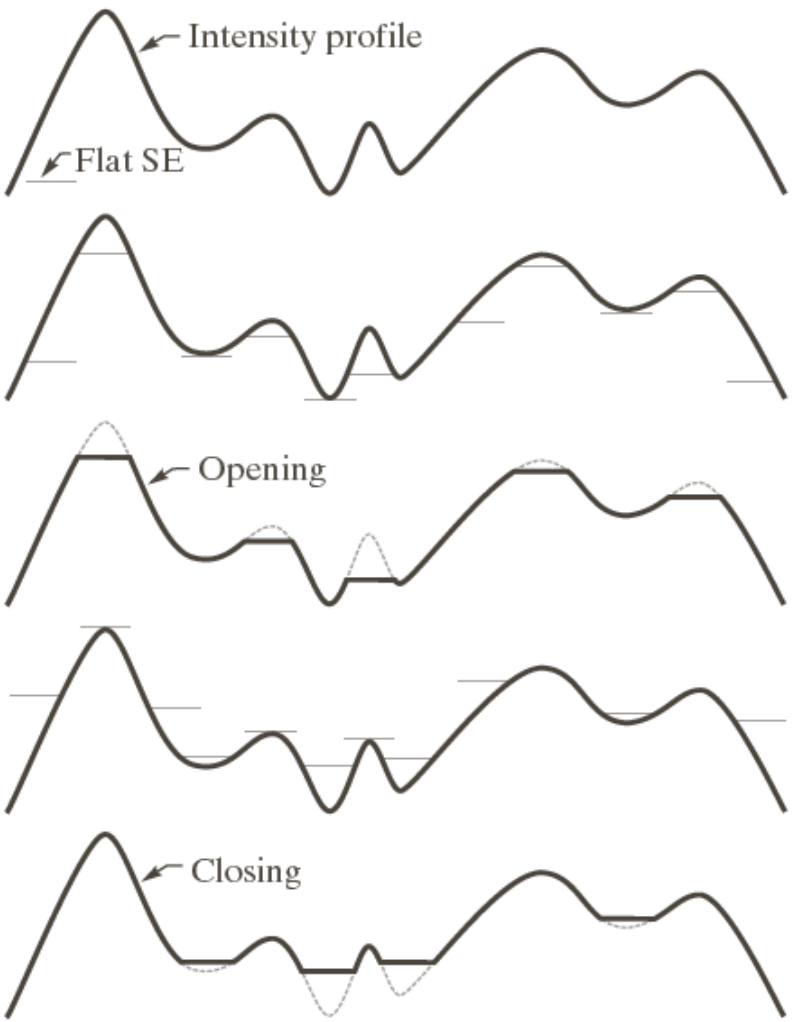
Dilation using flat disk SE with a radius of 2 pixels

Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Gray-Scale Opening & Closing

- Opening $$f \circ b = (f \ominus b) \oplus b$$
- Closing $$f \bullet b = (f \oplus b) \ominus b$$
- Duality $$(f \bullet b)^c = f^c \circ b$$
$$-(f \bullet b) = -(f \circ b)$$
- Properties
$$(f \circ b) \circ b = f \circ b \qquad (f \bullet b) \bullet b = f \bullet b$$

Gray-Scale Opening & Closing



a
b
c
d
e

FIGURE 9.36 Opening and closing in one dimension. (a) Original 1-D signal. (b) Flat structuring element pushed up underneath the signal. (c) Opening. (d) Flat structuring element pushed down along the top of the signal. (e) Closing.

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Gray-Scale Opening & Closing

- Opening
  - All the peaks that were narrow with respect to the diameter of the ball were reduced in amplitude and sharpness
  - It is applied to remove small light details, while leaving the overall gray levels and larger bright features undisturbed
- Closing
  - It is generally used to remove dark details, while leaving bright features undisturbed

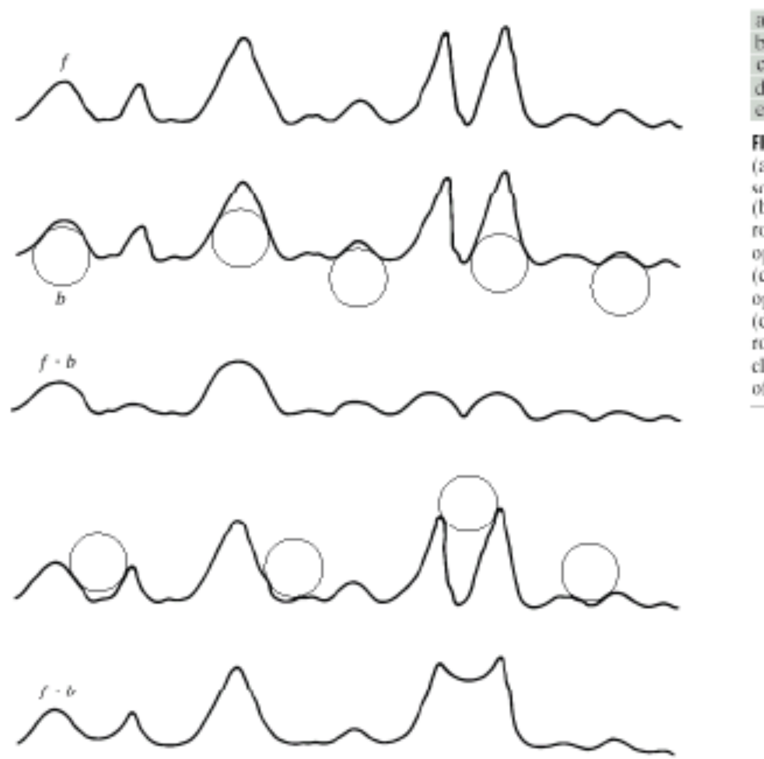

FIGURE 9.30 (a) A gray-scale scan line. (b) Positions of rolling ball for opening. (c) Result of opening. (d) Positions of rolling ball for closing. (e) Result of closing.

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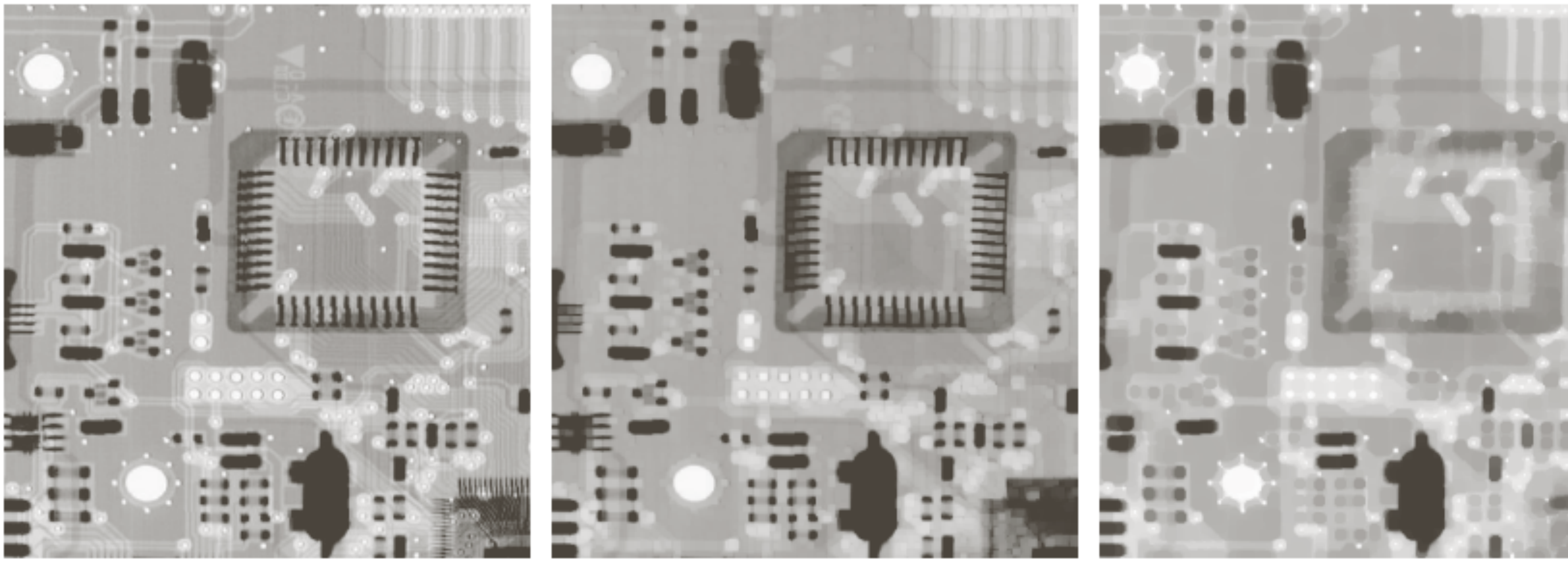
Gray-Scale Opening & Closing



a b c

FIGURE 9.37 (a) A gray-scale X-ray image of size 448 × 425 pixels. (b) Opening using a disk SE with a radius of 3 pixels. (c) Closing using an SE of radius 5.

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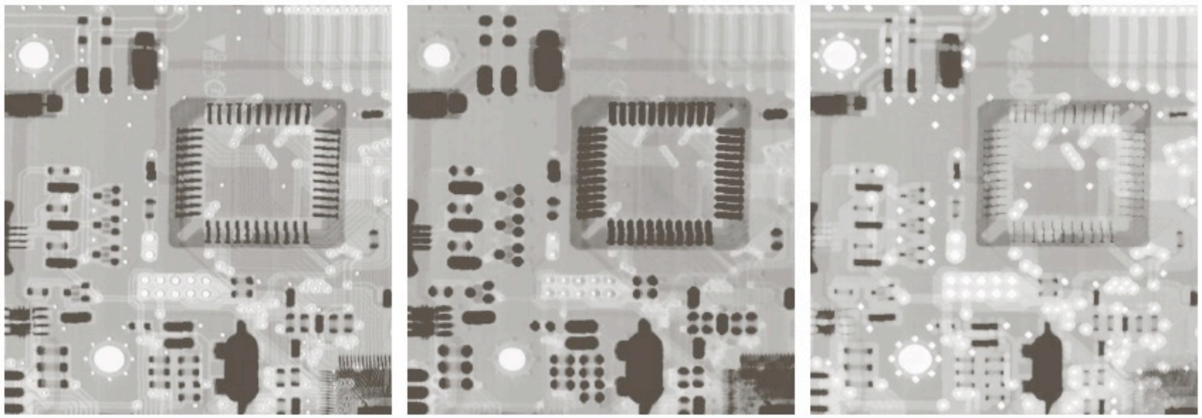
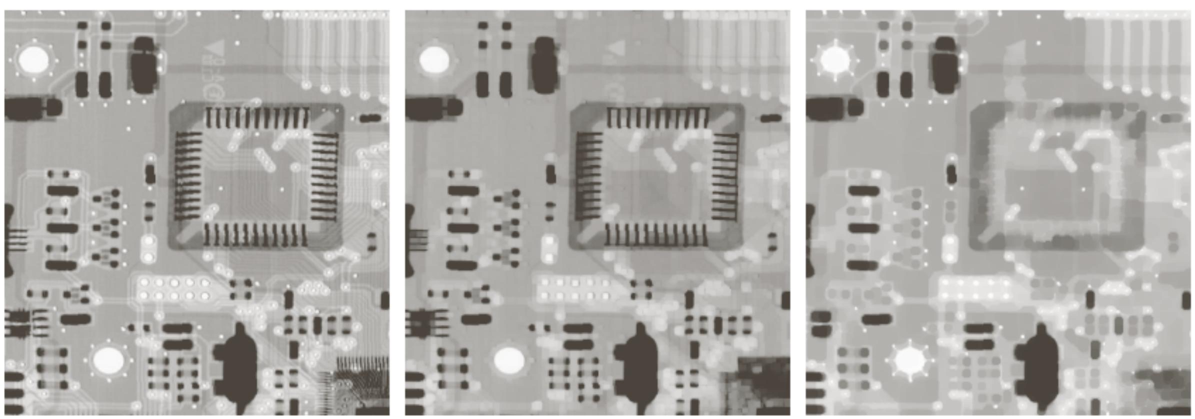
Comparison: Erosion & Dilation / Opening & Closing



Erosion / Dilation



Opening / Closing

Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Gray-Scale Opening & Closing

a b

FIGURE 9.31 (a) Opening and (b) closing of Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

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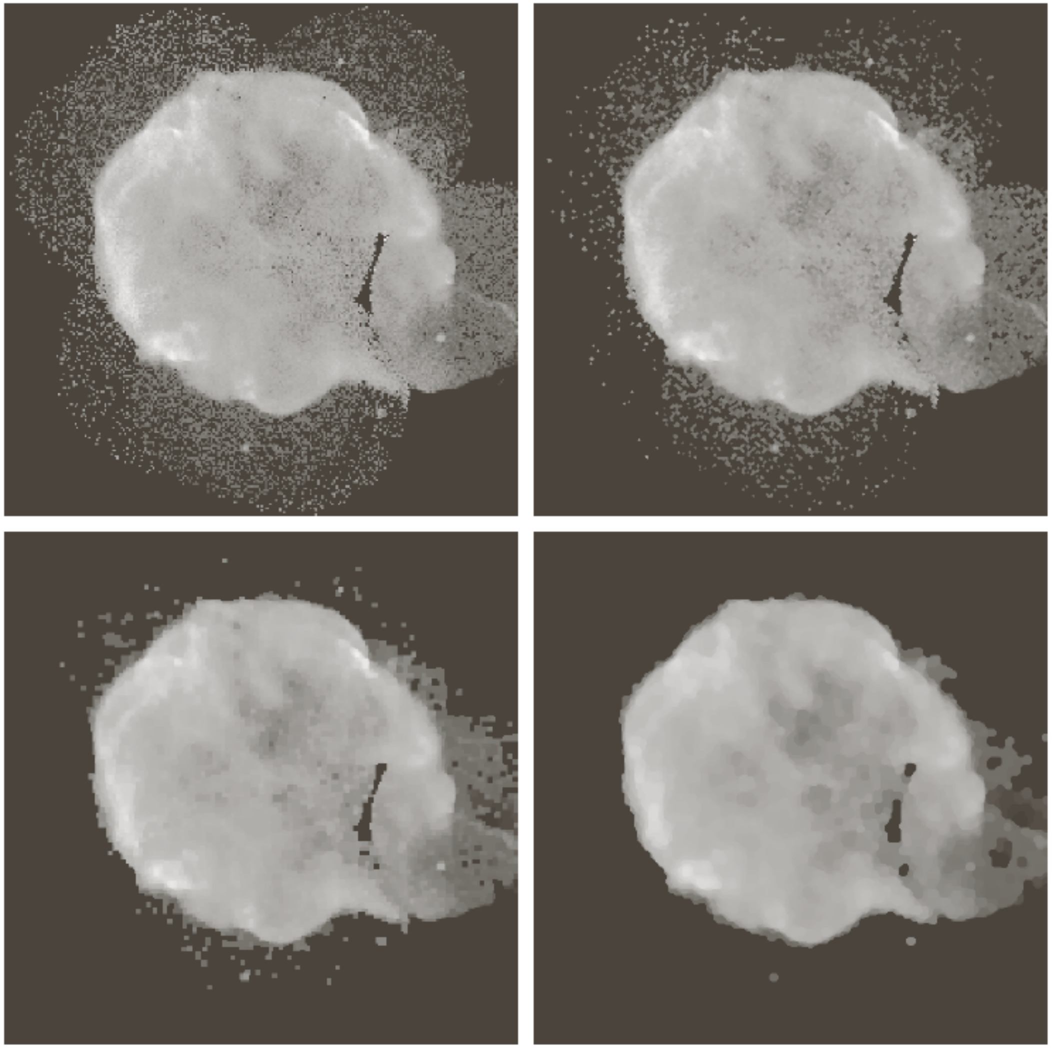
Morphological Smoothing

- Smoothing: Opening or Closing
- Remove or Attenuate both bright as well as dark artifacts or noise
- Called: Morphological Filters

Morphological Smoothing



a b
c d

FIGURE 9.38 (a) 566 × 566 image of the Cygnus Loop supernova, taken in the X-ray band by NASA's Hubble Telescope. (b)–(d) Results of performing opening and closing sequences on the original image with disk structuring elements of radii, 1, 3, and 5, respectively. (Original image courtesy of NASA.)

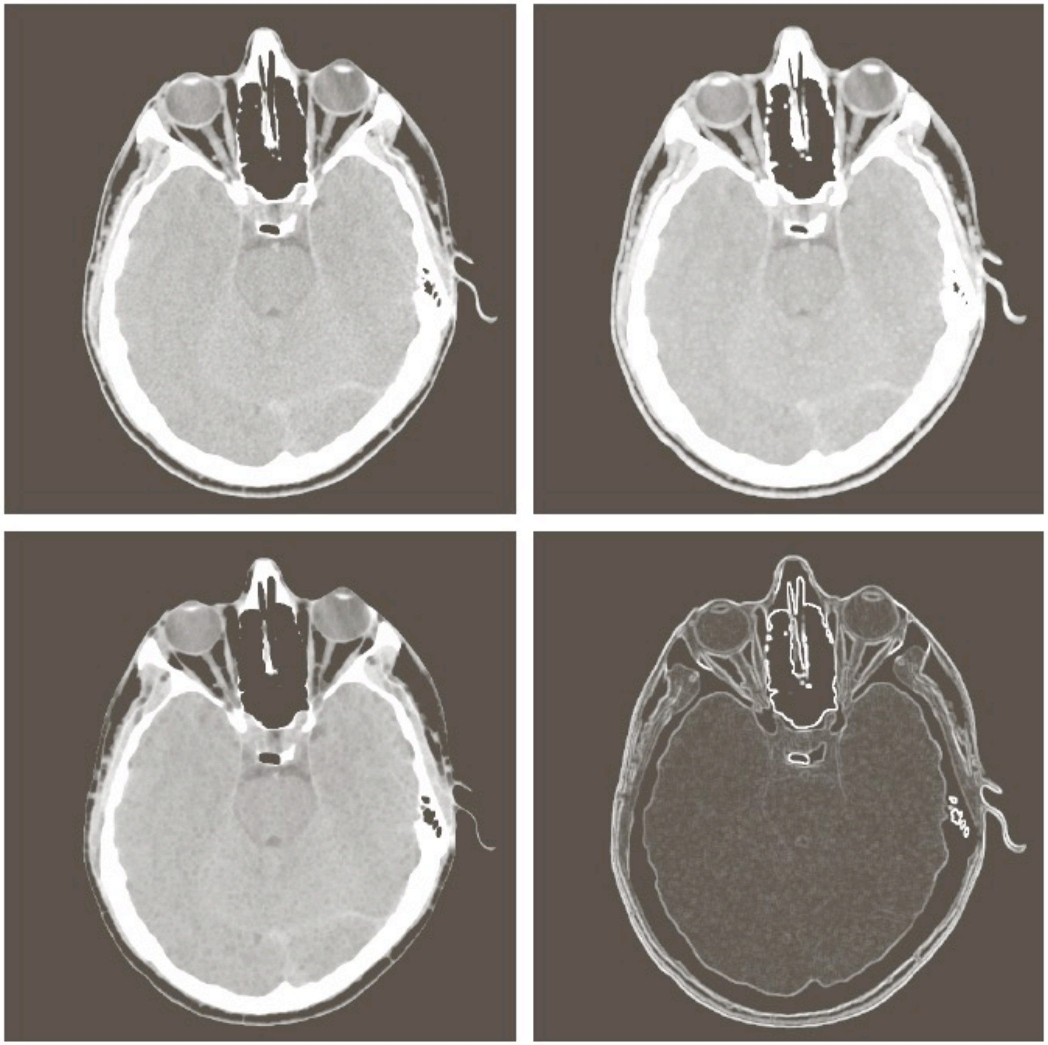
Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Morphological Gradient

$$g = (f \oplus b) - (f \ominus b)$$

- Highlights sharp gradient transition.
- Usually symmetrical SE used to make it less sensitive of edge direction.

Morphological Gradient



a	b
c	d

FIGURE 9.39 (a) 512 × 512 image of a head CT scan. (b) Dilation. (c) Erosion. (d) Morphological gradient, computed as the difference between (b) and (c). (Original image courtesy of Dr. David R. Pickens, Vanderbilt University.)

Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Top-Hat / Bottom-Hat Transformations

- Removing objects from an image
  - SE should not fit the object size

$$T_{hat}(f) = f - (f \circ b)$$

$$B_{hat}(f) = (f \bullet b) - f$$

- Top-Hat: Light objects on dark background
- Bottom-Hat: Dark objects on light background

Top-Hat Transformation

- The radius of the disk of the SE > width of the objects of interest.
- Resulting operation suppresses the background and makes objects prominent.

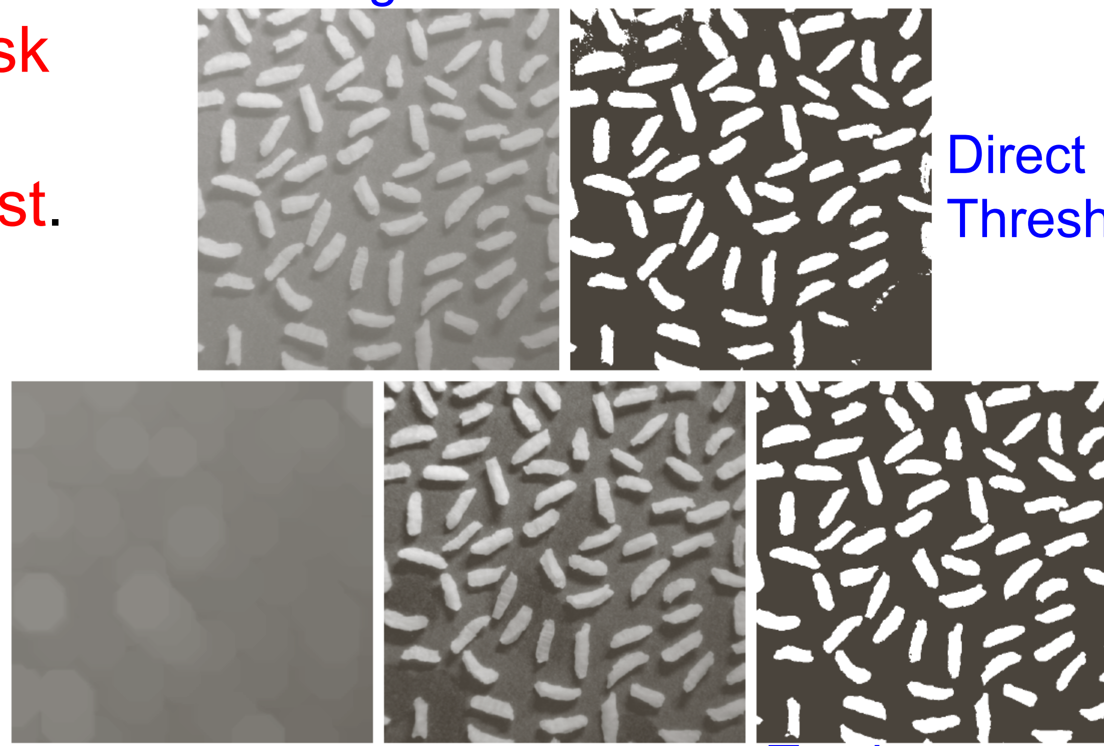


FIGURE 9.40 Using the top-hat transformation for *shading correction*. (a) Original image of size 600 × 600 pixels. (b) Thresholded image. (c) Image opened using a disk SE of radius 40. (d) Top-hat transformation (the image minus its opening). (e) Thresholded top-hat image.

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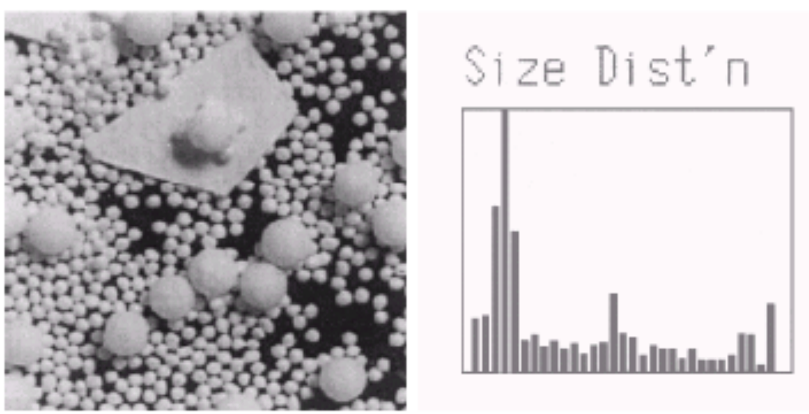
Granulometry

- Granulometry is a field that deals principally with determining the size distribution of particles in an image
- Opening operations with structuring elements of increasing size are performed on the original image
- The difference between the original image and its opening is computed after each pass
- At the end of the process, these differences are normalized and then used to construct a histogram of particle-size distribution



a b

FIGURE 9.36 (a) Original image consisting of overlapping particles; (b) size distribution. (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

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Granulometry

- A field principally dealing with determining the distribution of size of particles in an image.
- Opening with SEs of increasing size performed with the original image and the differences noted.
- These differences normalized and used to construct a histogram of the particle size distribution.

Granulometry

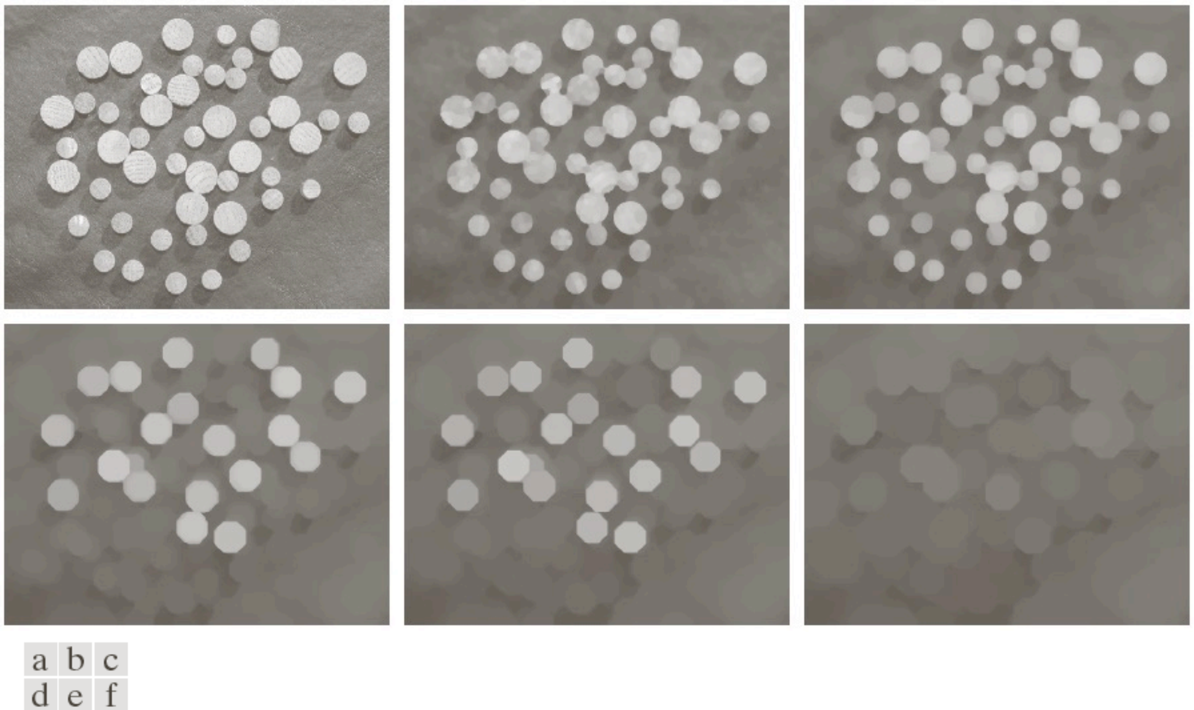

FIGURE 9.41 (a) 531 × 675 image of wood dowels. (b) Smoothed image. (c)–(f) Openings of (b) with disks of radii equal to 10, 20, 25, and 30 pixels, respectively. (Original image courtesy of Dr. Steve Eddins, The MathWorks, Inc.)

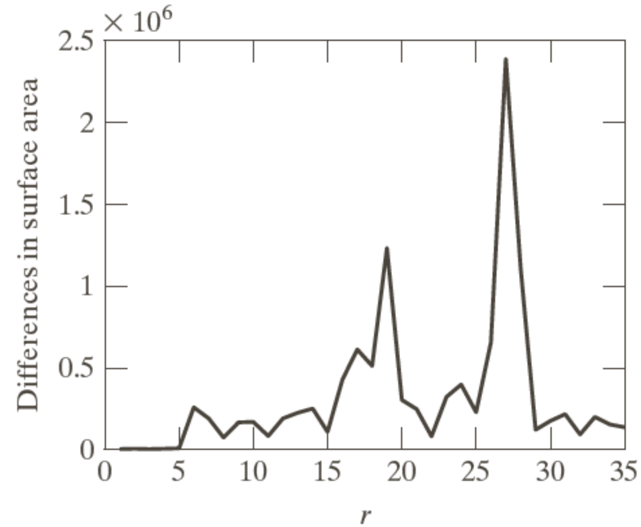

FIGURE 9.42 Differences in surface area as a function of SE disk radius, r. The two peaks are indicative of two dominant particle sizes in the image.

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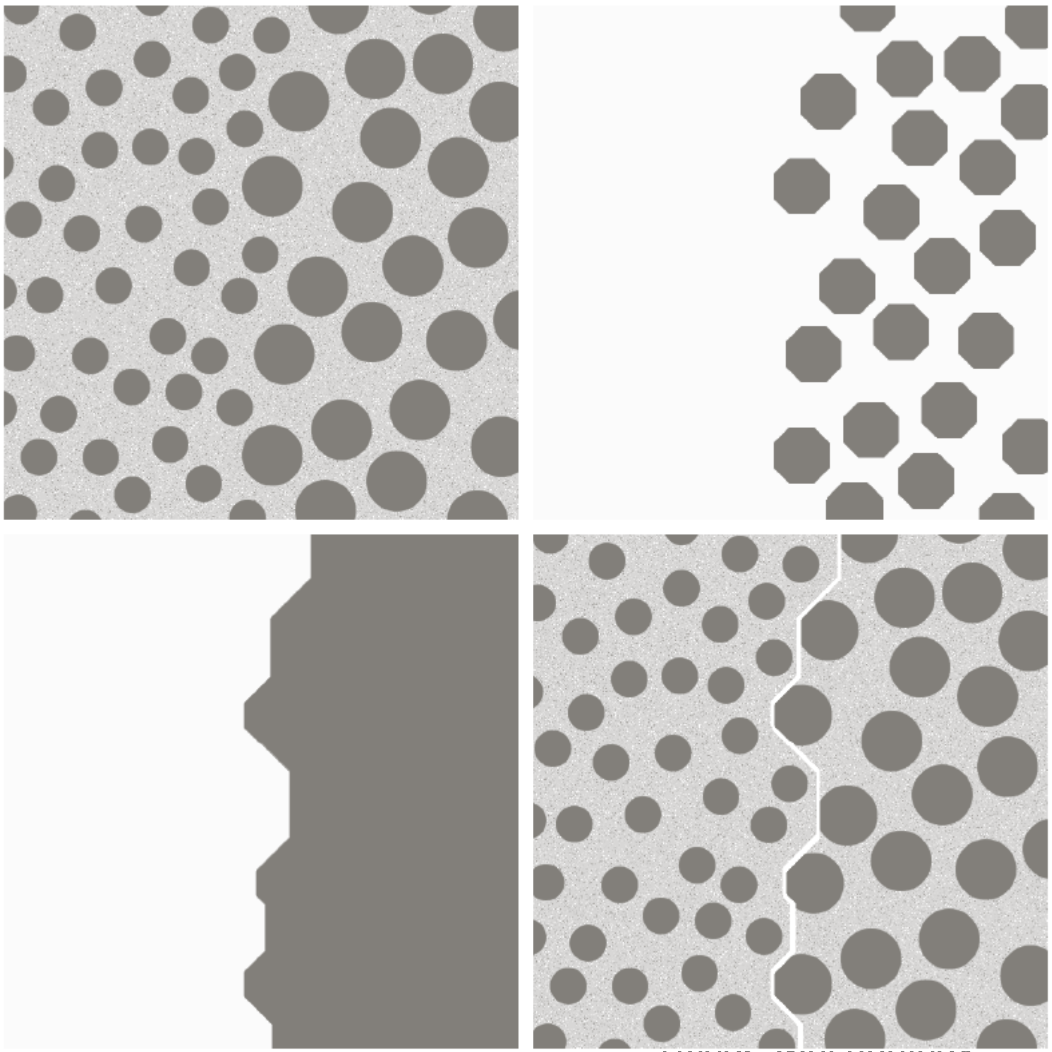
Textural Segmentation

- To find the boundary between two regions of different sizes of blobs.
- Closing with SEs of increasing size
  - When the size of SE is the same as a group of blobs, they are removed.
- A single opening with the SE of size larger than the separation between the larger blobs.
  - Dark or background regions
- A simple thresholding provides the boundary.

Textural Segmentation



a b
c d

FIGURE 9.43 Textural segmentation. (a) A 600 × 600 image consisting of two types of blobs. (b) Image with small blobs removed by closing (a). (c) Image with light patches between large blobs removed by opening (b). (d) Original image with boundary between the two regions in (c) superimposed. The boundary was obtained using a morphological gradient operation.

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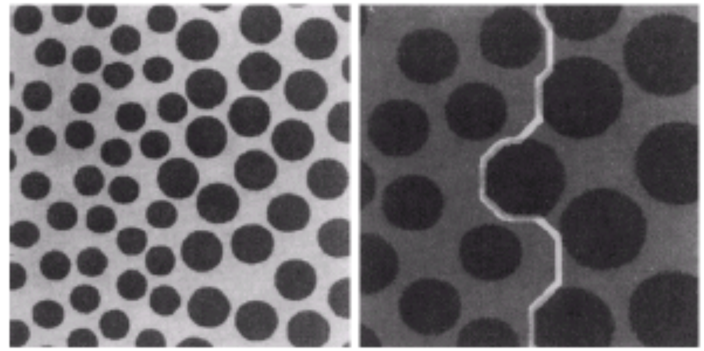
Textural Segmentation



a b

FIGURE 9.35 (a) Original image. (b) Image showing boundary between regions of different texture. (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

- The objective is to find the boundary between the two regions based on their textural content
- Closing with successively larger structuring elements

  → When the size of the structuring element corresponds to that of the small blobs, they are removed
- A single opening is performed with a structuring element that is larger than the separation between the large blobs → dark region on the right
- Then, a simple thresholding yields the boundary

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Summary

- An image f(x,y) treated as a function over space.
- The functional values can be treated as
  - A random variable, a random field, discretization of a continuous function
  - Pixel mapping, contrast enhancement, gradient computation, sharpening, correlation, convolution, linear and nonlinear filters
- Gray level morphology deals with structural analysis of an image with a pattern as a SE
  - Dilation, erosion, opening, closing, top-hat transformation
  - Gradient computation, smoothing, granulometry, textural segmentation

Thank You