Image Transforms and Compression

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Image Transform

$$f(x,y) = \sum_j \sum_i \lambda_{ij} b_{ij}(x,y)$$

- Image in continuous form: $f(x,y)$: A 2-D function, where (x,y) in R^2.

Properties of basis functions can be extended in the analysis.

- Let B be a set of basis functions:

$B=\{b_i(x,y) \mid i=..,-1,0,1,2,3,\ldots.\}$, $b_i(x,y)$ in R or C.

- Let $f(x,y)$ be expanded using B as follows:

$$f(x,y) = \sum_i \lambda_i b_i(x,y)$$

Coefficients of transform

The **transform** of f w.r.t. B is given by $\{\lambda_i | i = \cdots.-1,0,1,2,3,\ldots\}$.

Indexing may be multidimensional say, λ_{ij}.

Orthogonal Expansion and 1-D Transforms

$$f(x) = \sum_i \lambda_i b_i(x)$$

- Inner product: $$< f, g >= \int f(x) g^*(x) dx$$

- Orthogonal expansion: If B satisfies :

$$\langle b_i, b_j \rangle = 0,\ for\ i \neq j$$
$$= c_i\ Otherwise\ (for\ i = j), where\ c_i > 0$$

- Transform coefficients in O.E.:
  - c_i=1→ orthonormal expansion.

$$\lambda_i = \frac{1}{c_i} \langle f, b_i \rangle$$

Forward transform

$$\lambda_i = \langle f, b_i \rangle$$

- Inverse transform:

$$f(x) = \int_{i=-\infty}^{\infty} \lambda_i b_i(x) di$$

Fourier transform

Complete base

$$B = \{e^{jωx} | -\infty < \omega < \infty\}$$

Unit impulse function

Orthogonality: $$\int_{-\propto}^{\propto} e^{j\omega x} dx = \begin{cases} 2\pi\delta(x), & for\ \omega = 0 \\ 0, & otherwise. \end{cases}$$

Fourier Transform: $$\mathcal{F}(f(x)) = \hat{f}(j\omega) = \int_{-\propto}^{\propto} f(x) e^{-j\omega x} dx$$

Inverse Transform: $$f(x) = \frac{1}{2\pi} \int_{-\propto}^{\propto} \hat{f}(j\omega)\, e^{j\omega x} d\omega$$

Full reconstruction

$$e^{-j\omega x} = \cos(\omega x) - j \sin(\omega x)$$

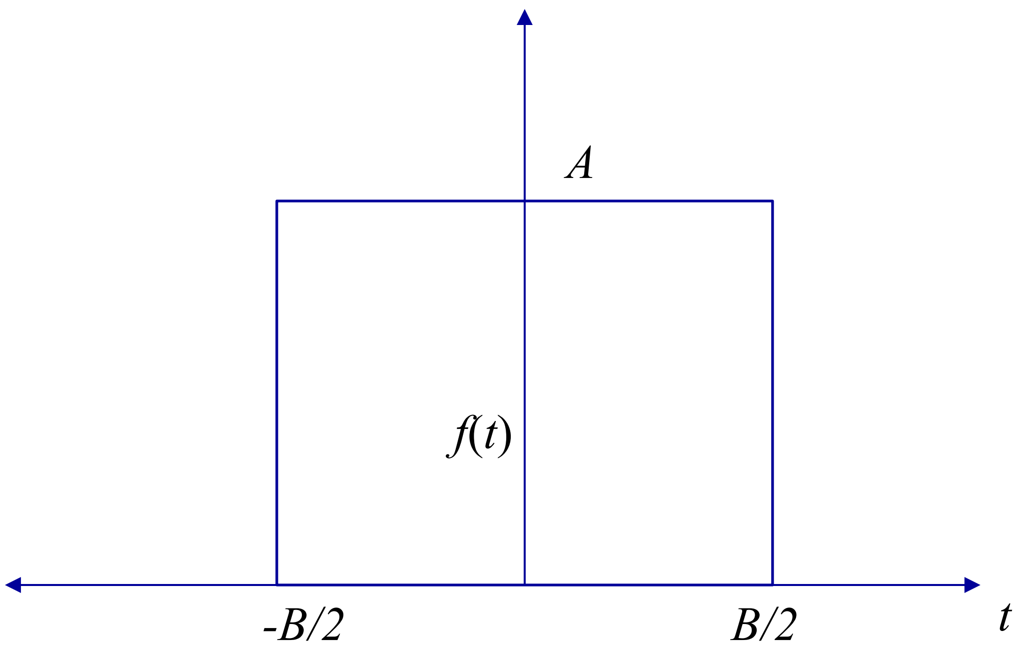
$$\hat{f}(j\omega) = \int_{-\infty}^{\infty} f(x)(\cos(\omega x) - j \sin(\omega x))\, dx$$

$$C = \{\cos(\omega x) | -\infty < \omega < \infty\}$$ $$S = \{\sin(\omega x) | -\infty < \omega < \infty\}$$

Orthogonal But not complete!

Fourier transform of a square pulse

- $f(t) = A, \ -B/2 \leq t \leq B/2$



$sinc(\omega B/2)$

$$F(j\omega) = AB\,\frac{\sin(\frac{\omega B}{2})}{\frac{\omega B}{2}}$$

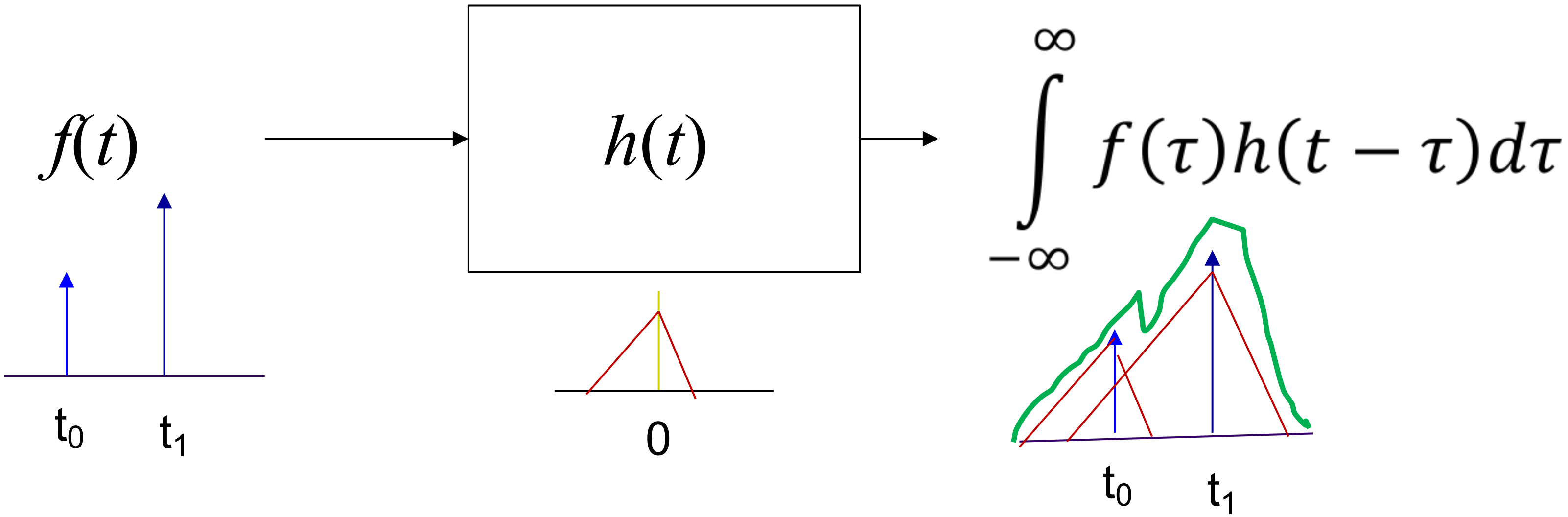
Fourier transform of a Gaussian Pulse

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

$$G(\omega) = e^{-\frac{\omega^2\sigma^2}{2}}$$

- Transform is also a Gaussian function.
- Standard deviation in the Fourier domain (angular frequency) is reciprocal of that in the time domain.

Convolution and Fourier Transform



- Convolved output: Sum of scaled and shifted impulse responses.

$$F\big(f(t) * h(t)\big) = \int \Big(\int f(\tau)\, h(t-\tau) d\tau \Big)\, e^{-j\omega t} dt \quad = \int f(\tau) \int h(t-\tau) e^{-j\omega t} dt\, d\tau$$

$$= \int f(\tau)\, H(j\omega) e^{-j\omega\tau} d\tau$$

$$= H(j\omega) F(j\omega)$$

Fourier transform of unit impulse

- Definition and properties of unit infinite impulse

$$\delta(t) = \infty, t = 0$$
$= 0,\quad$ otherwise

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

$$F(\delta(t)) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt \qquad = e^{-j\omega .0} = 1$$

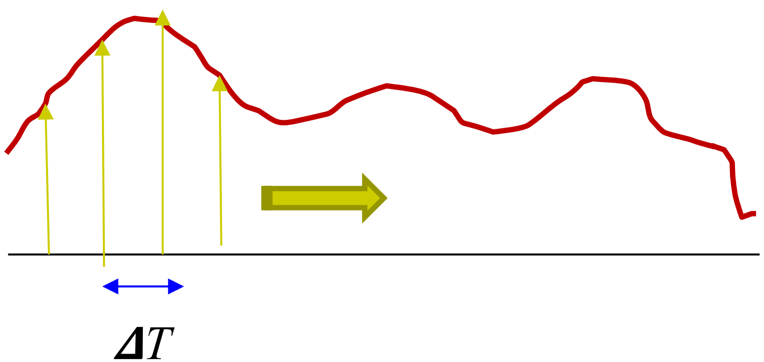
$$F(\delta(t - T)) = e^{-j\omega T} \quad \Longleftrightarrow \quad F\left(e^{j\omega_0 t}\right) = \delta(\omega - \omega_0)$$

Duality

Train of impulses: $\sum \delta(t - n\Delta T) \xrightarrow{\mathcal{F}} \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{\Delta T})$

Fourier series of a period ΔT
with unit impulse in a period

Fourier transform of a sampled function



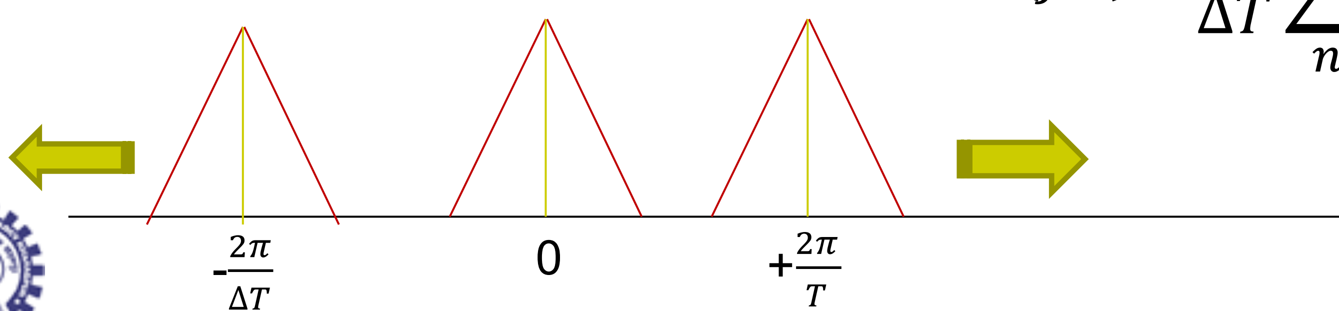
$$f_s(t) = \sum f(t)\delta(t - n\Delta T)$$

$$= f(t)\sum \delta(t - n\Delta T)$$

- $f_s(t) = f(t)$, for $t = n\,\Delta T$, n: an integer.

  $= 0$, otherwise

$$F(j\omega) * \frac{1}{\Delta T}\sum_{n} \delta(\omega - \frac{2\pi n}{\Delta T})$$



Even and odd functions

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- Even: $f(-x)=f(x)$ for all x.
- Odd: $f(-x)=-f(x)$ for all x. $\rightarrow f(0)=0$.
- For even $f(x)$:

$$\int_{-\infty}^{\infty} f(x)(\sin(\omega x))\, dx = 0$$

- For odd $f(x)$:

$$\int_{-\infty}^{\infty} f(x)(\cos(\omega x))\, dx = 0$$

- Full reconstruction possible with cosines (sines) only if it is even (odd).

Discrete representation

- Discrete representation of a function:

$$f(n) = \{f\ (nX_0) | n \epsilon \mathbb{Z}\}$$

Set of integers

Sampling interval

- Can be considered as a vector in an infinite dimensional vector space.
- In our context, it is of a finite dimensional space, e.g. $\{f(n),\ n=0,1,..N-1\}$, or
- $\boldsymbol{f}=[f(0)\ f(1)\ \ldots\ f(N-1)]^T$.

Discrete Linear Transform: A general form

- For n-dimensional vector X any linear transform,
  - e.g. $Y_{m\text{x}1}=B_{m\text{x}n}X_{n\text{x}1}$
  - $X_{n\text{x}1}$: *A column vector of dimension n.*
  - $Y_{m\text{x}1}$: *A column vector of dimension m.*
  - $B_{m\text{x}n}$: *A matrix of dimension m*x*n.*
- Has inverse transform if B is a square matrix and invertible.

Basis vectors

- B is the transformation matrix.
- Rows of B are called basis vectors.

$$B = \begin{bmatrix} \boldsymbol{b}_0^{*T} \\ \boldsymbol{b}_1^{*T} \\ \vdots \\ \boldsymbol{b}_n^{*T} \end{bmatrix}$$

- $Y(i) = <\boldsymbol{b}_i^{*T}.\ X>$

dot product or inner product.

- Orthogonality condition:

$$\begin{aligned} < \boldsymbol{b}_i^{*T} . \boldsymbol{b}_j > &= 0 \ if \ i \neq j \\ &= c_i, \quad\quad otherwise \end{aligned}$$

Discrete Fourier Transform (DFT)

$$b_k(n) = \frac{1}{\sqrt{N}} e^{j2\pi\frac{k}{N}n}, \text{ for } 0 \le n \le N-1, \text{ and } 0 \le k \le N-1.$$

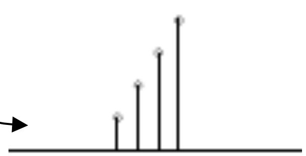
$F(k)$ ⟺ $$\hat{f}(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi\frac{k}{N}n} \text{ for } 0 \le k \le N-1.$$ $\hat{f}(N+k) = \hat{f}(k)$

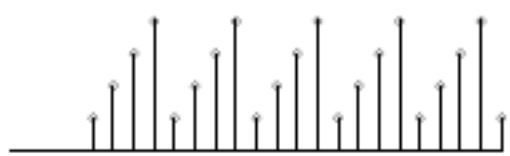
k/N: Normalized frequency

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}(k) e^{j2\pi\frac{k}{N}n} \text{ for } 0 \le n \le N-1.$$

Fundamental frequency: $1/(NX_0)$

A single period





$f(n+N)=f(n)$

DFT: Fourier series of a periodic function

DFT as Fourier Series

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}(k)e^{j2\pi\frac{k}{N}n} \text{ for } 0 \le n \le N-1.$$

$$\hat{f}(k) = \sum_{n=0}^{N-1} f(n)e^{-j2\pi\frac{k}{N}n} \text{ for } 0 \le k \le N-1.$$

- For a periodic sequence of period N: $f(n+N)=f(n)$
- Sampling interval: X_0
- Period: $N \,.\, X_0$
  - Fundamental period: $1/(NX_0)$
  - Fourier series: Components of $k/(NX_0)$, k=0,1,2,…

$$F\left(\frac{k}{NX_0}\right) = \frac{1}{NX_0}\sum_{n=0}^{N-1} f(nX_0)\, e^{-j2\pi\frac{k}{NX_0}nX_0}\Delta x$$

$$F(k) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\, e^{-j2\pi\frac{k}{N}n}$$

In the DFT expression the normalization term (1/N) is adjusted.

For any integer k, $e^{j2\pi k} = 1$

DFT properties

- Linearity: $DFT\,(a.f(n) + b(g(n))) = aF(k) + bG(k)$
- Circular time shifting $DFT(f(\langle n - n_0\rangle_N)) = e^{-j2\pi\frac{k}{N}n_0}F(k)$
- Periodicity:

$$F(k+N) = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi\frac{k+N}{N}n} = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi(\frac{k}{N}+1)n} = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi\frac{k}{N}n} = F(k)$$

- Symmetry

$$F(N-k) = F(-k) = \sum_{n=0}^{N-1} f(n)\, e^{j2\pi\frac{k}{N}n} = F^*(k) \Rightarrow F\left(\frac{N}{2}+m\right) = F^*(\frac{N}{2}-m)$$

Putting, $k=N/2+m$

- Duality
  - DFT of DFT of $x(n) = N.\, x(<-k>_N)$
- Energy preservation

$$\vec{x}.\vec{y}^* = \frac{1}{N}\hat{\vec{x}}.\hat{\vec{y}}^* \Rightarrow \|\vec{x}\|^2 = \vec{x}.\vec{x}^* = \frac{1}{N}\hat{\vec{x}}.\hat{\vec{x}}^* = \frac{1}{N}\left\|\hat{\vec{x}}\right\|^2$$

- Freq. Shifting $DFT(f(n)e^{j2\pi\frac{k_0}{N}n}) = F(< k - k_0 >_N)$

Centering DFT

$$DFT\left(f(n)e^{j2\pi\frac{k_0}{N}n}\right) = F(< k - k_0 >_N)$$

- Multiplying k_0 th sinusoid shifts transform to k_0.
- Let $k_0=N/2$
  - ➔ $f(n)\,(-1)^n$
  - ➔ $F(<k - N/2>_N)$
  - Centers the Fourier transform bringing the 0 th freq. component in the center.
- A useful trick to center the transform
  - Multiply by $(-1)^n$ and then compute DFT.

Fast Fourier Transform (FFT)

$$F(k) = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi \frac{k}{N} n}$$

Assume N even

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)\, e^{-j2\pi \frac{k}{N}(2m)} + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)\, e^{-j2\pi \frac{k}{N}(2m+1)}$$

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)\, e^{-j2\pi \frac{k}{N/2}(m)} + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)\, e^{-j2\pi \frac{k}{\frac{N}{2}}(m+1/2)}$$

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)\, e^{-j2\pi \frac{k}{N/2}(m)} + e^{-j2\pi \frac{k}{\frac{N}{2}}(1/2)} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)\, e^{-j2\pi \frac{k}{\frac{N}{2}}(m)}$$

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)\, e^{-j2\pi \frac{k}{N/2}(m)} + e^{-j2\pi \frac{k}{N}} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)\, e^{-j2\pi \frac{k}{\frac{N}{2}}(m)}$$

Cooley, I.W., and Tukey, I.W., "An Algorithm for the Machine Calculation of Complex Fourier Series," Math. Comp., vol. 19, pp. 297-301, April 1965.

Fast Fourier Transform (FFT) (Cooley and Tukey (1965))

$$F(k) = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi \frac{k}{N} n}$$

Divide and conquer strategy

Exploiting other properties of DFT can be reduced to O(N log(N))

Assume N even

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m)\, e^{-j2\pi \frac{k}{N/2}(m)} + e^{-j2\pi \frac{k}{N}} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1)\, e^{-j2\pi \frac{k}{\frac{N}{2}}(m)}$$

DFT of order N/2 of Even terms

DFT of order N/2 of Odd terms

Danielson-Lanczos lemma

Danielson, G.C. and Lanczos, C., "Some Improvements In Practical Fourier Analysis and Their Application to X-Ray Scattering from Liquids," J. Franklin Institute, vol. 233, pp. 365-380 and 435-452, 1942.

$$\widehat{f \otimes h}(k) = \hat{f}(k)\hat{h}(k)$$

CMP for DFT

CMP for DFT holds for circular convolution.

$\boxed{\text{Linear convolution}}$

$$f(n) \longrightarrow \boxed{h(n)} \longrightarrow \sum_{m=-\infty}^{\infty} f(m)h(n-m)$$

- Periodic convolution: Convolution between two finite sequences with periodic extension.
- It is defined if both have the same period, providing a periodic sequence with the same period.

$\boxed{\text{Circular Convolution}}$

$$\begin{aligned} f \circledast h(n) &= \textstyle\sum_{m=0}^{N-1} f(m)h(n-m), \\ &= \textstyle\sum_{m=0}^{n} f(m)h(n-m) + \sum_{m=n+1}^{N-1} f(m)h(n-m+N). \end{aligned}$$

Circular Cross Correlation

- Cross correlation with periodic extensions of both the functions.

$$f \odot h(n) = \sum_{m=0}^{N-1} f(m)\, h(m+n)$$

$$f \odot h(n) = \sum_{m=0}^{N-n-1} f(m)\, h(m+n) + \sum_{m=N-n}^{N-1} f(m)\, h(m+n-N)$$

$$\widehat{f \odot h}(k) = \hat{f}(k).\,\hat{h}(k)^*$$

Filtering in the transform domain

$$F(k) \longrightarrow \boxed{H(k)} \longrightarrow G(k)=H(k)\,F(k)$$

- Use sufficient 0 padding at the both end to make circular convolution equivalent to linear convolution
  - To take care of boundary effect.
  - The length of $f(n)$ and $h(n)$ should be the same.
- $H(k)$ usually provided as symmetric about the center.
  - 0^{th} freq. at the N/2 th element.
- Center $F(k)$ as $F_c(k)$ by multiplying f(n) with $(-1)^n$
- Obtain $G(k)= H(k)\,.\,F_c(k)$
- Multiply $G(k)$ by $(-1)^k$ and perform IDFT to get $g(n)$.

DFT: A linear transform

$$F(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi\frac{kn}{N}} \quad for\ 0 \le k \le N-1$$

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi\frac{1}{N}} & \cdots & e^{-j2\pi\frac{N-1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi\frac{N-1}{N}} & \cdots & e^{-j2\pi\frac{(N-1)^2}{N}} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}$$

$$\mathcal{F}_N = \left[e^{-j2\pi\frac{k}{N}n} \right]_{0 \le (k,n) \le N-1}$$

$$\boldsymbol{F} = \mathcal{F}_N \boldsymbol{f}$$

$$\boldsymbol{f} = \mathcal{F}_N^{-1} \boldsymbol{F}$$

Hermitian transpose

$$\mathcal{F}_N^{-1} = \frac{1}{N} \mathcal{F}_N^{H}$$

Generalized Discrete Fourier Transform (GDFT)

$$\mathbf{F}_{\alpha,\beta} = \left[e^{-j2\pi \frac{k+\alpha}{N}(n+\beta)} \right]_{0 \le (k,n) \le N-1}.$$

$$\begin{aligned}
\mathbf{F}_{0,0}^{-1} &= \tfrac{1}{N}\mathbf{F}_{0,0}^{H} &= \tfrac{1}{N}\mathbf{F}_{0,0}^{*}, \\
\mathbf{F}_{\frac{1}{2},0}^{-1} &= \tfrac{1}{N}\mathbf{F}_{\frac{1}{2},0}^{H} &= \tfrac{1}{N}\mathbf{F}_{0,\frac{1}{2}}^{*}, \\
\mathbf{F}_{0,\frac{1}{2}}^{-1} &= \tfrac{1}{N}\mathbf{F}_{0,\frac{1}{2}}^{H} &= \tfrac{1}{N}\mathbf{F}_{\frac{1}{2},0}^{*}, \text{ and} \\
\mathbf{F}_{\frac{1}{2},\frac{1}{2}}^{-1} &= \tfrac{1}{N}\mathbf{F}_{\frac{1}{2},\frac{1}{2}}^{H} &= \tfrac{1}{N}\mathbf{F}_{\frac{1}{2},\frac{1}{2}}^{*}.
\end{aligned}$$

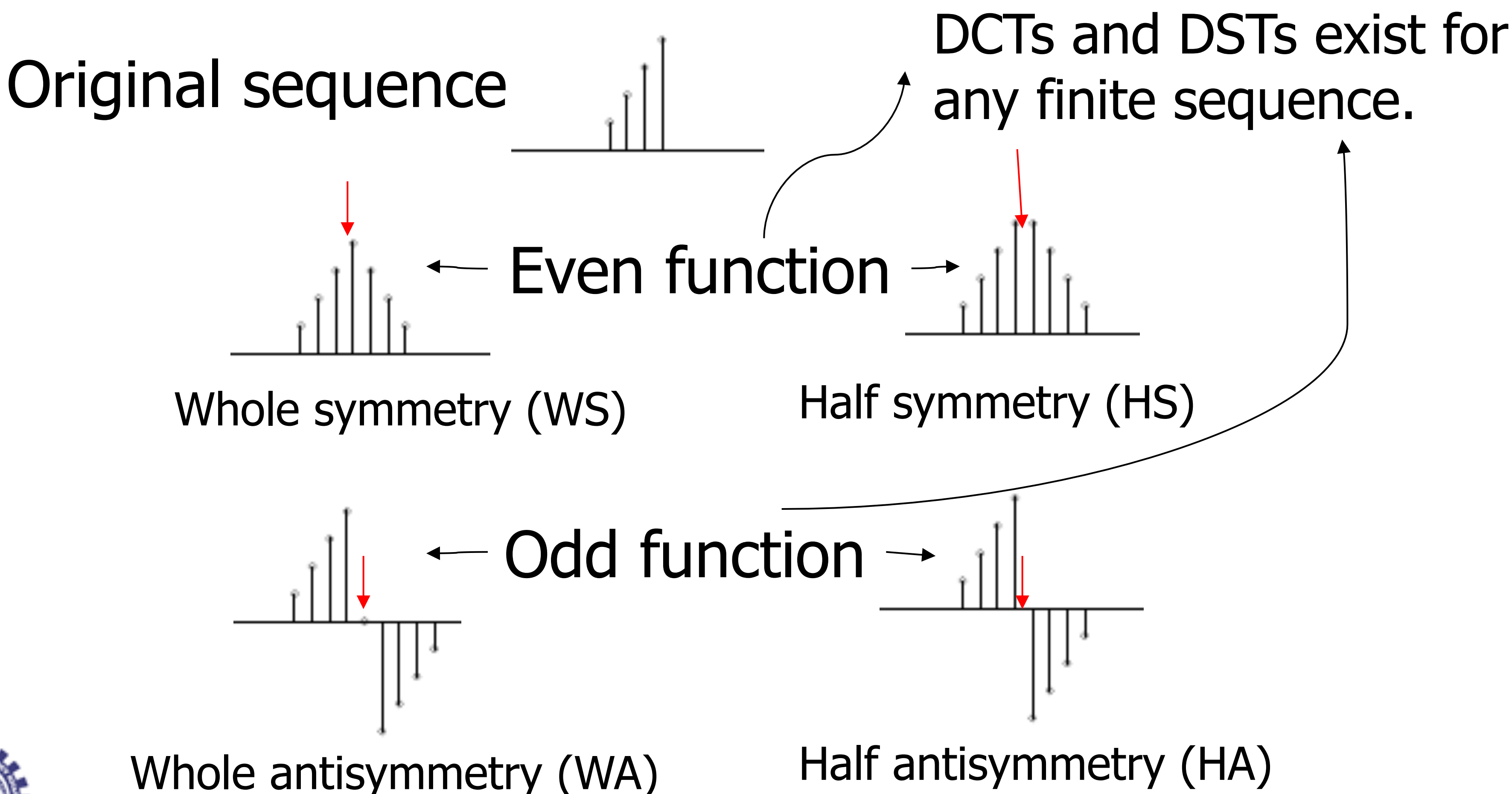
$$b_k^{(\alpha,\beta)}(n) = \frac{1}{\sqrt{N}} e^{j2\pi \frac{k+\alpha}{N}(n+\beta)}, \text{ for } 0 \le n \le N-1, \text{ and } 0 \le k \le N-1$$

$$\hat{f}_{\alpha,\beta}(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi \frac{k+\alpha}{N}(n+\beta)}, \text{ for } 0 \le k \le N-1$$

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}_{\alpha,\beta}(k) e^{j2\pi \frac{k+\alpha}{N}(n+\beta)}, \text{ for } 0 \le n \le N-1$$

α	β	Transform name	Notation
0	0	Discrete Fourier Transform (DFT)	$\hat{f}(k)$
0	$\frac{1}{2}$	Odd Time Discrete Fourier Transform ($OTDFT$)	$\hat{f}_{0,\frac{1}{2}}(k)$
$\frac{1}{2}$	0	Odd Frequency Discrete Fourier Transform ($OFDFT$)	$\hat{f}_{\frac{1}{2},0}(k)$
$\frac{1}{2}$	$\frac{1}{2}$	Odd Frequency Odd Time Discrete Fourier Transform (O^2DFT)	$\hat{f}_{\frac{1}{2},\frac{1}{2}}(k)$

Symmetric / Antisymmetric extension of a finite sequence

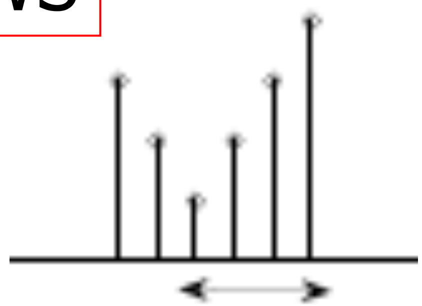


Discrete Cosine / Sine Transforms

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

- Types of symmetric / antisymmetric extensions at the two ends of a sequence and a type of GDFT→ DCTs / DSTs

$\boxed{\text{WSWS}}$



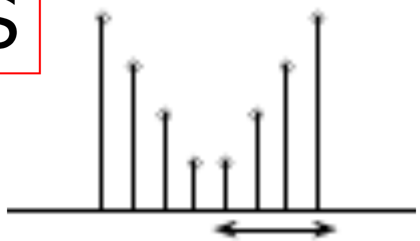
$\xrightarrow{F_{0,0}}$ Type-I Even DCT

$$C_{1e}(x(n)) = X_{Ie}(k) = \sqrt{\frac{2}{N}}\alpha^2(k)\sum_{n=0}^{N} x(n)\cos\left(\frac{2\pi nk}{2N}\right),\ 0 \le k \le N,$$

Discrete Cosine / Sine Transforms

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

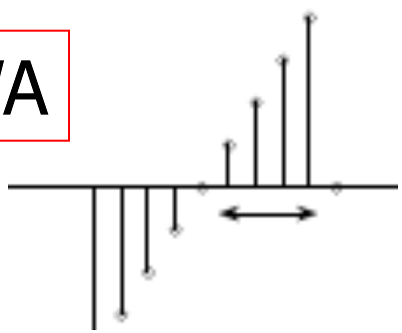
HSHS



$F_{0,1/2}$ $\Longrightarrow$ Type-2 Even DCT

$$C_{2e}(x(n)) = X_{IIe}(k) = \sqrt{\frac{2}{N}}\alpha(k)\sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi k(n+\frac{1}{2})}{2N}\right),\ 0 \le k \le N-1$$

WAWA



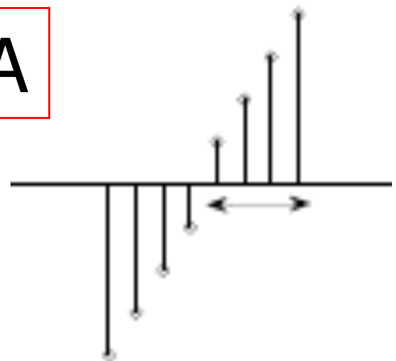
$jF_{0,0}$ $\Longrightarrow$ Type-1 Even DST

$$S_{1e}(x(n)) = X_{sIe}(k) = \sqrt{\frac{2}{N}}\sum_{n=1}^{N-1} x(n)\sin\left(\frac{2\pi kn}{2N}\right),\ 1 \le k \le N-1$$

Discrete Cosine / Sine Transforms

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

HAHA



$\xrightarrow{jF_{0,1/2}}$ Type-2 Even DST

$$S_{2e}(x(n)) = X_{sIIe}(k) = \sqrt{\frac{2}{N}}\alpha(k)\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi k(n+\frac{1}{2})}{2N}\right),\ 1 \le k \le N-1$$

There exist 16 different types of DCTs and DSTs.

Type-II Even DCT is used in signal, image, and video compression.

Matrix form of Type-II DCT

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

- Matrix form:

N-point DCT

$$C_N = \left[\sqrt{\frac{2}{N}}.\alpha(k)cos(\frac{\pi k(2n+1)}{2N}) \right]_{0 \le (k,n) \le N-1} .$$

- Each row is either symmetric (even row) or antisymmetric (odd row).

$$C_N(k, N-1-n) = \begin{cases} C_N(k,n) & for\ k\ \ even \\ -C_N(k,n) & for\ k\ \ odd \end{cases}$$

$$X = C_N.\mathbf{x} \qquad\qquad C_N^{-1} = C_N^T$$

Antiperiodic extension and skew-circular convolution

- Antiperiodic function with an antiperiod N, if $f(x+N)=-f(x)$.
- An antiperiodic function of antiperiod N → a periodic function of period $2N$.
- Skew-circular convolution: convolution between two antiperiodic extended sequences of the same antiperiod.

$$\begin{aligned} f \circledS h(n) &= \sum_{m=0}^{N-1} f(m)h(n-m), \\ &= \sum_{m=0}^{n} f(m)h(n-m) - \sum_{m=n+1}^{N-1} f(m)h(n-m+N) \end{aligned}$$

CMPs for DCTs

$$u(n) = x(n) \circledast y(n)$$
$$w(n) = x(n) \circledS y(n)$$

$$C_{1e}(u(n)) = \sqrt{2N} C_{1e}(x(l)) C_{1e}(y(m))$$

$$C_{2e}(u(n)) = \sqrt{2N} C_{2e}(x(l)) C_{1e}(y(m))$$

$$C_{3e}(w(n)) = \sqrt{2N} C_{3e}(x(l)) C_{3e}(y(m))$$

$$f(x,y)=\sum_j \sum_i \lambda_{ij} b_{ij}(x,y)$$

2-D Transforms

- Easily extendable if basis functions are separable, i.e. $B=\{ b_{ij}(x,y)=g_i(x).g_j(y)\}$.

They could be from two different sets, say $b(x,y)=g(x).h(y)$.

1-D basis function

- B: Orthogonal if $G=\{g_i(x), i=1,2,..\}$ is orthogonal.
- B: Orthogonal and complete if G is so.
- Reuse of 1-D transform computation.

$$\lambda_{ij}=\sum_j g_j^*(y)\left(\sum_i f(x,y) g_i^*(x)\right)$$

2D Discrete Transform

$$Y_{m\text{x}n} = B_{m\text{x}m} X_{m\text{x}n} B_{n\text{x}n}^{\mathrm{T}}$$

- Use of separability:
  - Transform columns.
  - Transform rows.
- Input: $X_{m\text{x}n}$ 1-D Transform Matrix: *B*
- Transform columns: $[Y_1]_{m\text{x}n} = B_{m\text{x}m} X_{m\text{x}n}$
- Transform rows: $Y_{m\text{x}n} = [B_{n\text{x}n} Y_1^{\mathrm{T}}]^{\mathrm{T}}$

$$= Y_1 B_{n\text{x}n}^{\mathrm{T}}$$

$$= B_{m\text{x}m} X_{m\text{x}n} B_{n\text{x}n}^{\mathrm{T}}$$

Image Transform: DFT

Image: $f(m,n)$, of size M x N

$$F(k,l)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)e^{-j2\pi\frac{km}{M}}e^{-j2\pi\frac{ln}{N}}$$

$$f(m,n)=\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}F(k,l)e^{j2\pi\frac{km}{M}}e^{j2\pi\frac{ln}{N}}$$

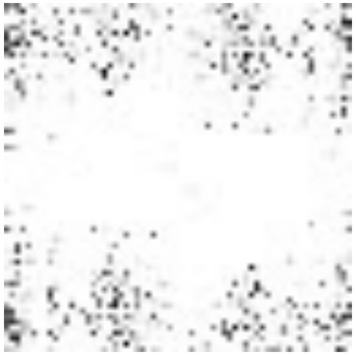
Property of separability

$$\boldsymbol{F}=\mathcal{F}_m\boldsymbol{f}\mathcal{F}_N^T$$

$$F(k,l)=\sum_{m=0}^{M-1}e^{-j2\pi\frac{km}{M}}\sum_{n=0}^{N-1}f(m,n)e^{-j2\pi\frac{ln}{N}}$$

DFT Examples:

Magnitude



Phase

Magnitudes and phases are shown by bringing them into displayable range, and shifting the origin at the center of image.

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

2D DCT

- Type-I:

$$X_I(k,l) = \tfrac{2}{N}.\alpha^2(k).\alpha^2(l).\textstyle\sum_{m=0}^{M}\sum_{n=0}^{N}(x(m,n)\cos(\frac{m\pi k}{M})\cos(\frac{n\pi l}{N})), \quad 0 \le k \le M, 0 \le l \le N.$$

- Type-II

- Matrix Representation:

$$X_{II}(k,l) = \tfrac{2}{N}.\alpha(k).\alpha(l).\textstyle\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(x(m,n)\cos(\frac{(2m+1)\pi k}{2M})\cos(\frac{(2n+1)\pi l}{2N})), \quad 0 \le k \le M-1, 0 \le l \le N-1.$$

$$X = DCT(x) = C_M.\mathrm{x}.C_N^T$$

An example:

Input image

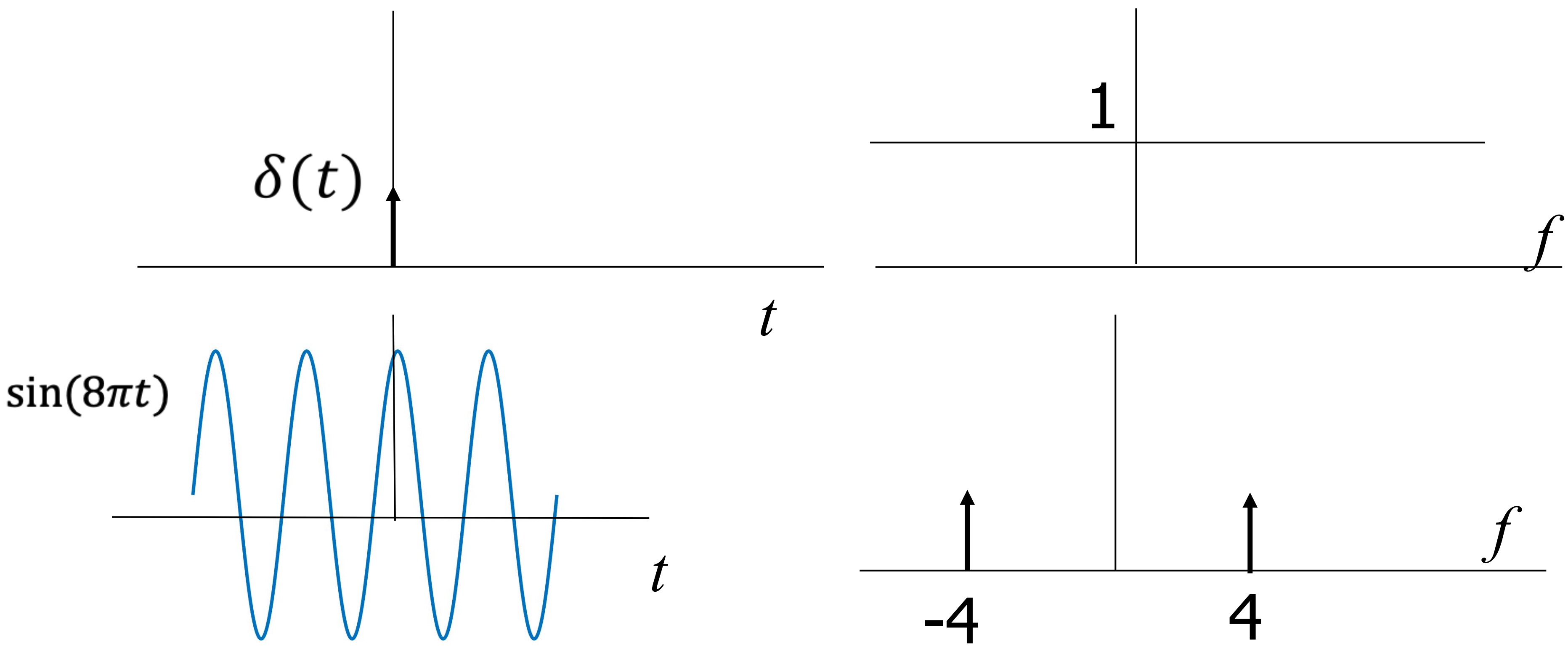
Discrete Cosine Transform

There are 16 different types of DCTs and DSTs.

Wavelets

- Functions to have *ideally* finite support in both its original domain (say, time or space) and also in the transform domain (i.e., the frequency domain).
  - No such function exists truly satisfying it.
  - Attempts to match these properties as far as possible.
- Acts as basis functions.
- Good localization property in both domain.

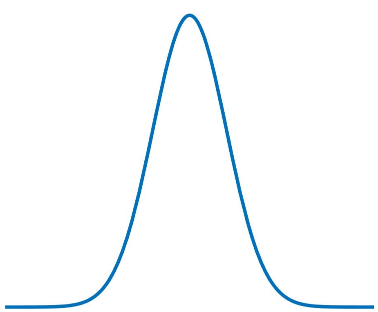
A few examples



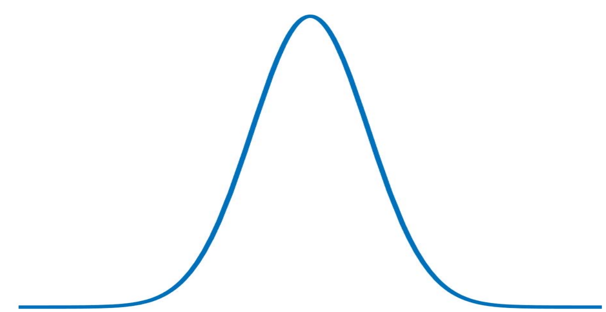
An interesting function

- Same form in time and frequency domain
  - Gaussian
- Analogy from Heisenberg's uncertainty principle

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{4}$$



Variance of t weighted by $g^2(t)$. Similarly for f. For any function it holds !!



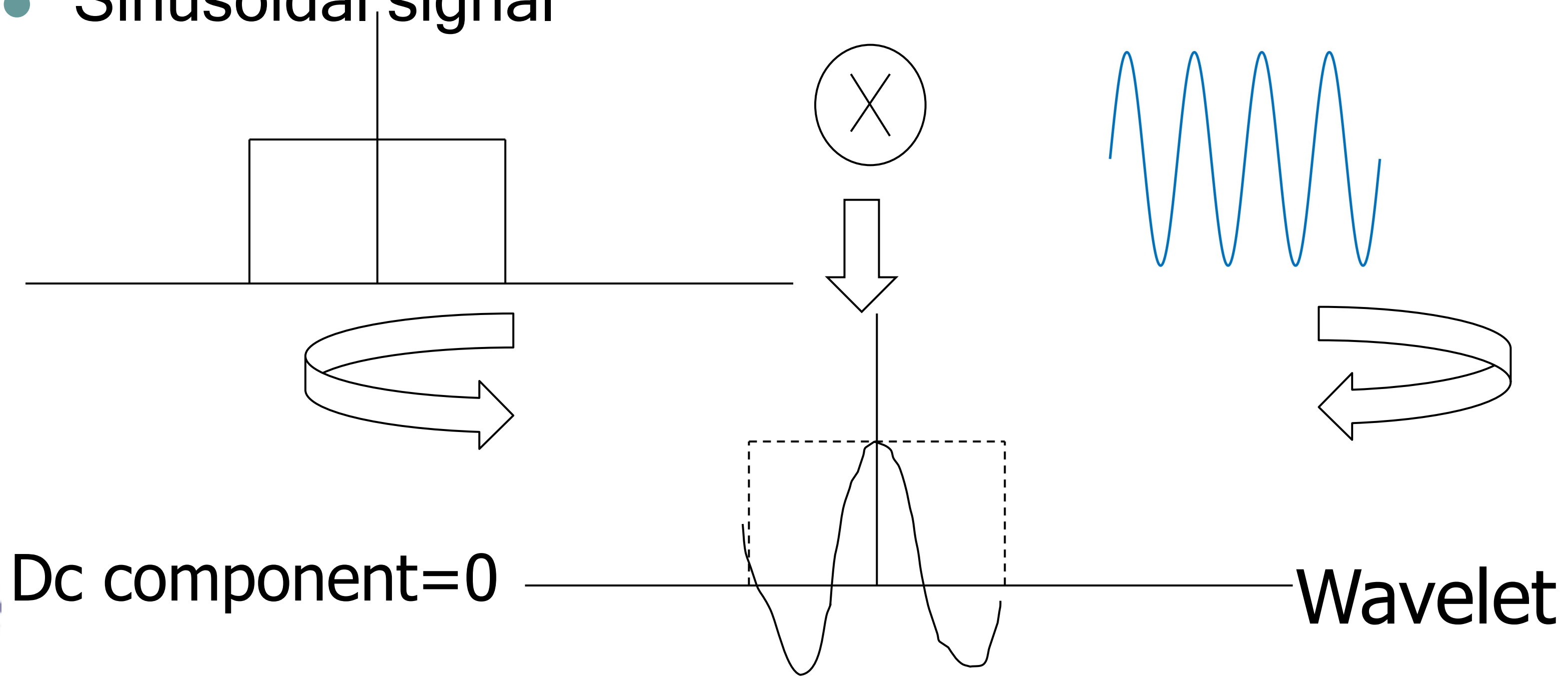
$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

$$G(\omega) = e^{-\frac{\omega^2\sigma^2}{2}}$$

Designing wavelet: An intuitive approach

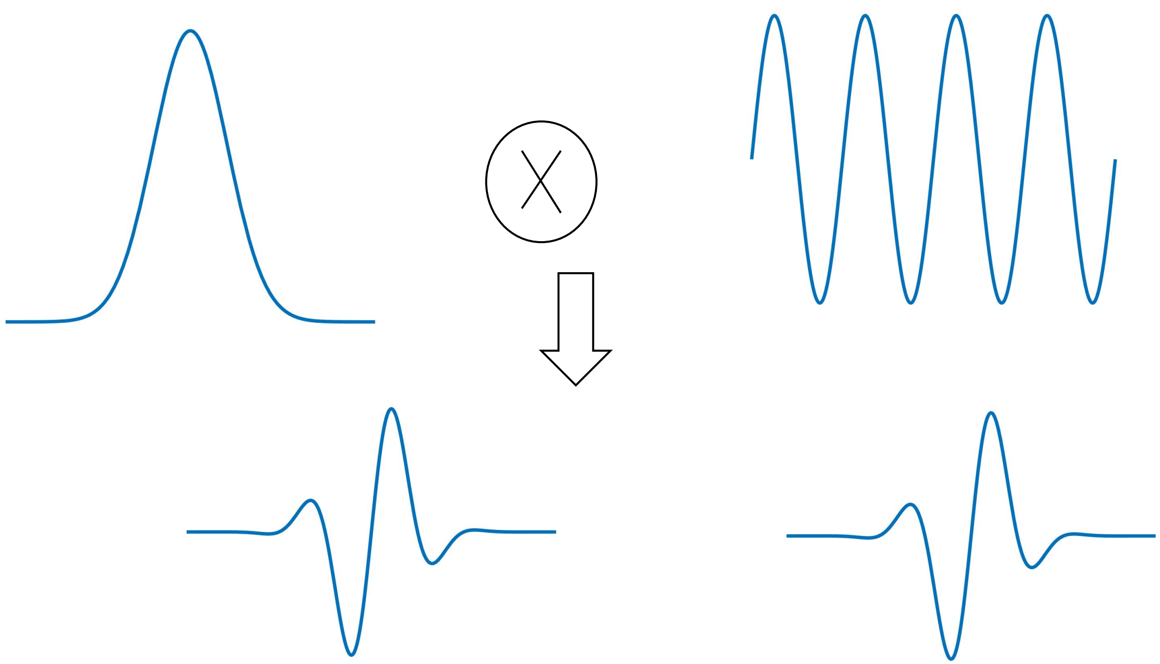
- Time limited signal:
  - Square pulse
- Band limited signal:
  - Sinusoidal signal

- Wavelet to satisfy both?
  - Multiply them!!



Gabor wavelet (1-D)

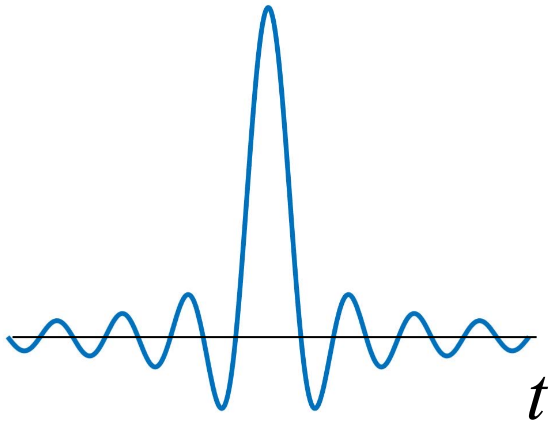
$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \qquad e^{j2\pi f t}$$

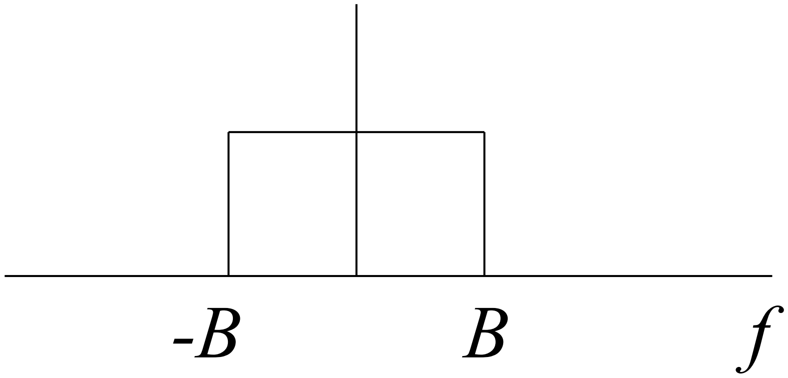


Real part Imaginary part

Shannon wavelet

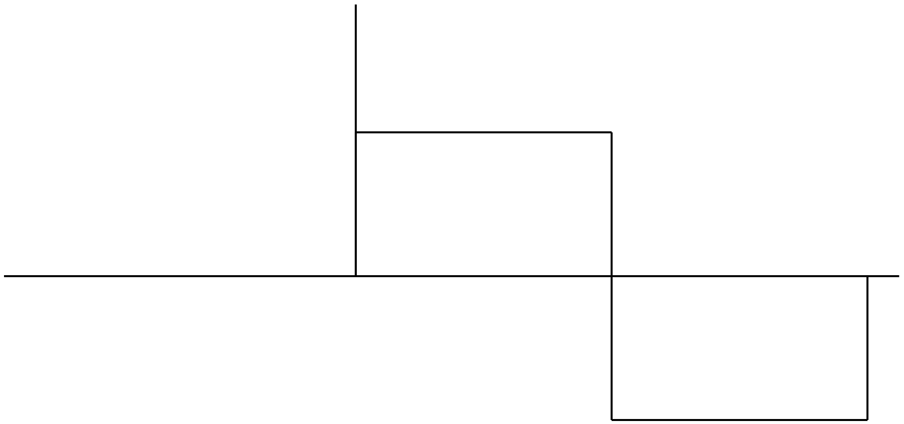
$$h(t) = 2B\,\frac{sin(2\pi Bt)}{2\pi Bt} = 2B sinc(2Bt)$$





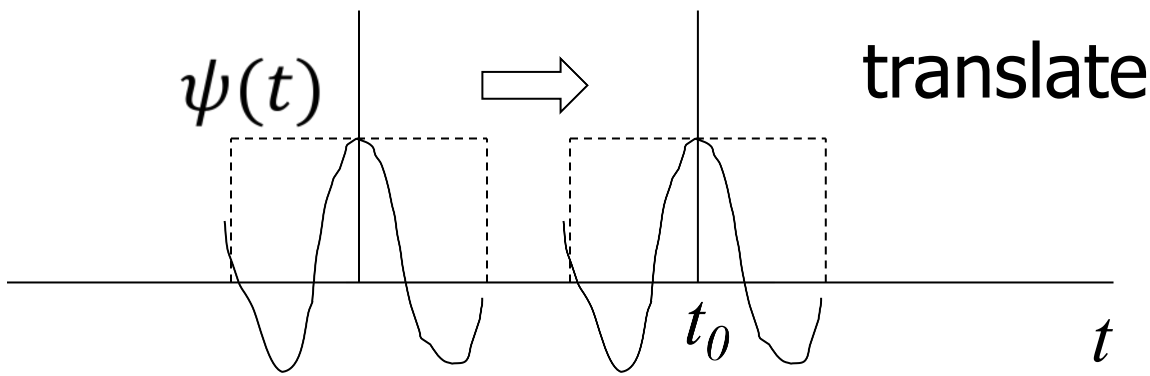
Haar Wavelet

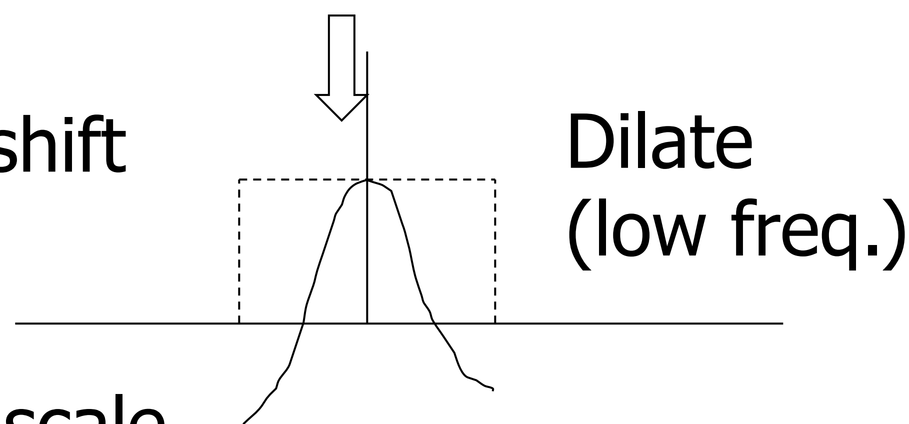
$$\psi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} < t \leq 1 \\ 0 & Otherwise \end{cases}$$



Family of wavelets

- Translate and dilate a mother wavelet





shift

$$\psi_{\tau,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

scale

Continuous wavelet transform

- Forward transform

From 1-D representation to 2-D representation.

$$W(s,\tau) = \int f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-\tau}{d}\right)dt$$

How correlated at that instance with the wavelet fn.

Reveals structure of function at multiple resolution.

- Inverse transform:

$$f(t) = \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty W(s,\tau)\frac{\psi(t)}{s^2}\,ds\,d\tau$$

Fourier transform

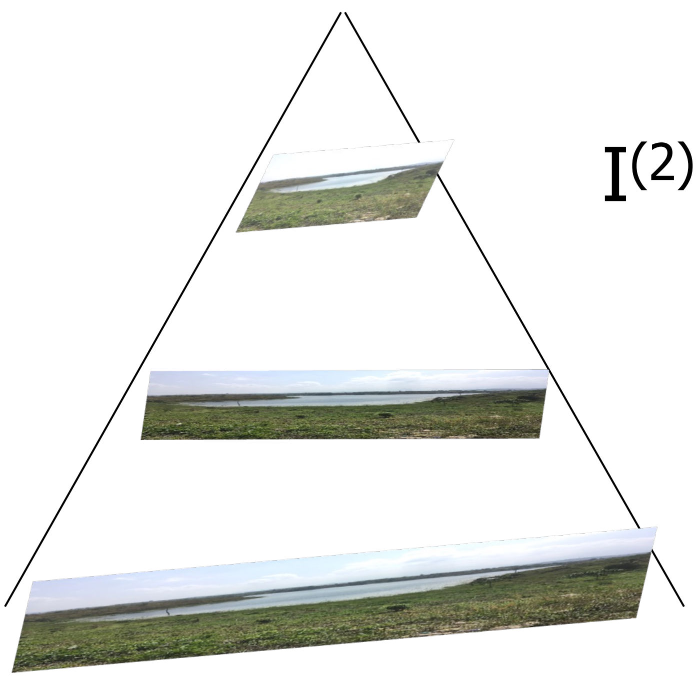
where

$$C_\psi = \int_{-\infty}^\infty \frac{|\hat{\psi}(\omega)|}{|\omega|}d\omega$$

Multiresolution representation

- Gaussian Pyramid

$d^{(2)} = I^{(1)} - \text{upsampled}(I^{(2)})$



$I^{(2)}$ $\oplus$ $d^{(2)}$

$I^{(1)}$ $\oplus$ $d^{(1)}$

$I^{(0)}$

$d^{(1)} = I^{(0)} - \text{upsampled}(I^{(1)})$

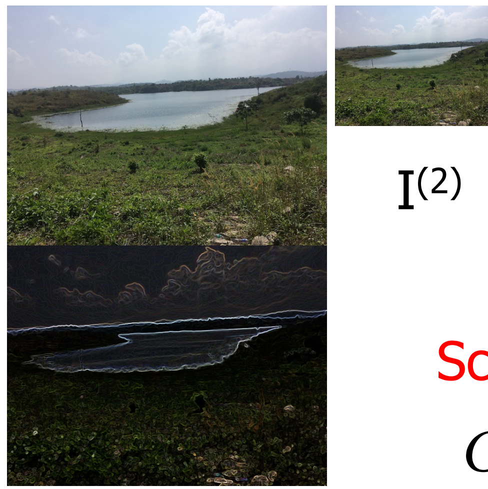
$d^{(i)} = I^{(i-1)} - \text{upsampled}(I^{(i)})$

$I(0) = us(us(I^{(2)}) + d^{(2)}) + d^{(1)}$

Approximation

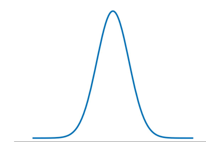
Details

Gaussian Pyramid: Wavelet analysis



$I^{(2)}$

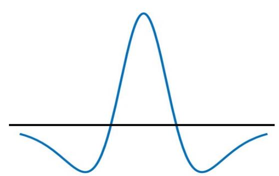
Obtained by convolution with G(x,y) and downsampling at successive stages.



Scaling function

$$G(x,y)=\frac{1}{2\pi\sigma^2}e^{\frac{-((x-x_c)^2+(y-y_c)^2)}{2\sigma^2}}$$

$d^{(2)}$



Obtained by convolution with DOG(x,y) and downsampling at successive stages.

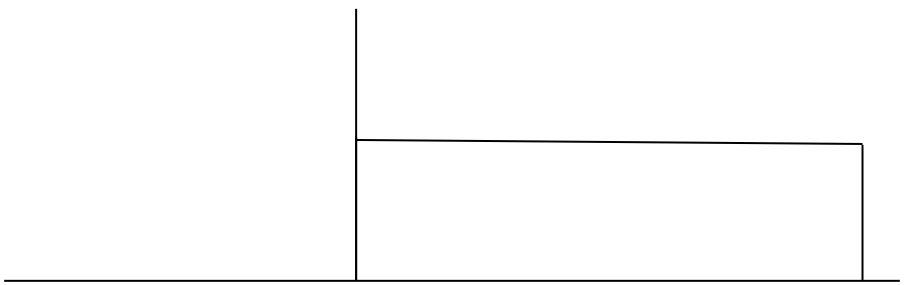
Wavelet function

$d^{(1)}$

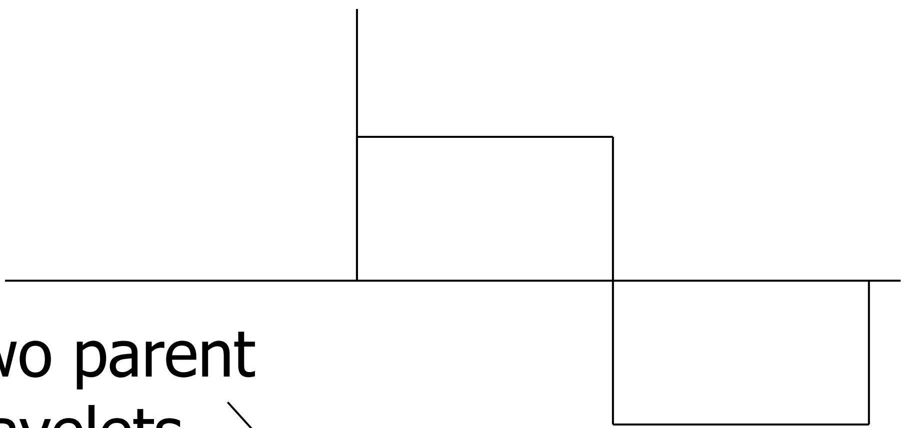
Filtering and transformation equivalent!!

Haar Wavelet transform

- Scaling function



- Wavelet function



Two parent wavelets

$$\varphi(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0 & Otherwise \end{cases}$$

$$\psi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} < t \leq 1 \\ 0 & Otherwise \end{cases}$$

Family of translated and dilated functions from the both forms the basis.

Discrete wavelet transform (DWT)

- Translated only at discrete grid points.
  - $k=0, \pm 1, \pm 2,$ ……
  - Finite sequence: A finite number of basis functions.
- Scaled by powers of 2: 2^j, $j=0,1,$…
  - Downsampling takes care of dilation of wavelets and allows to use the same function at that level.
- Family of scaling and wavelet functions:

$$\varphi_{j,k}(n) = 2^{-\frac{j}{2}}\varphi\left(2^{-j}n - k\right), j = 0,1, \ldots k = 0,1, \ldots, M$$

$$\psi_{j,k}(n) = 2^{\frac{-j}{2}}\psi\left(2^{-j}n - k\right), j = 0,1, \ldots k = 0,1, \ldots, M$$

M≤N (length of sequence)

Haar wavelets in discrete grid

- N=8

$$\varphi(n) = \frac{1}{\sqrt{2}}(1,1,0,0,0,0,0,0)$$

$$\psi(n) = \frac{1}{\sqrt{2}}(1,-1,0,0,0,0,0,0)$$

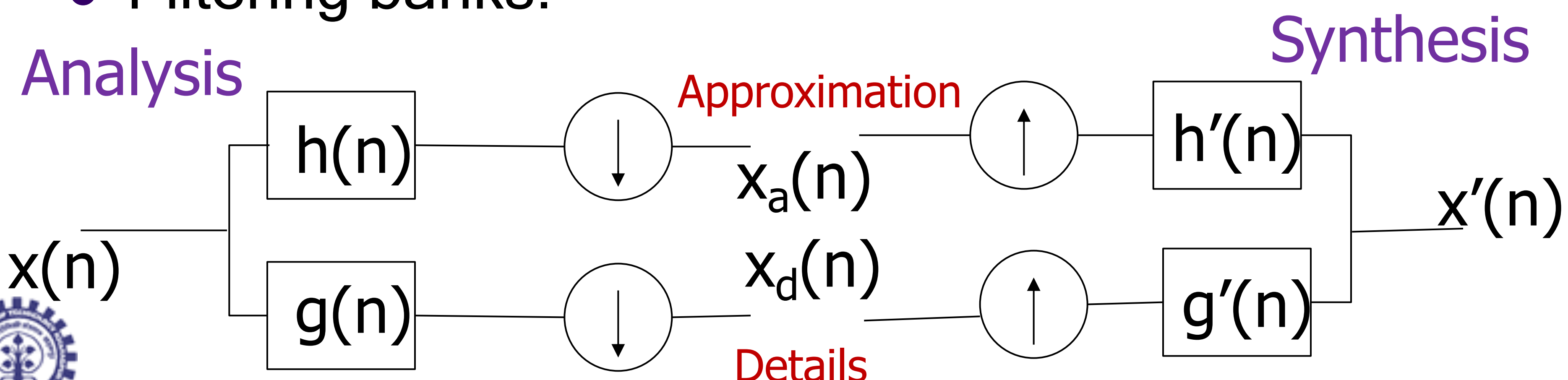
Transformation matrix:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

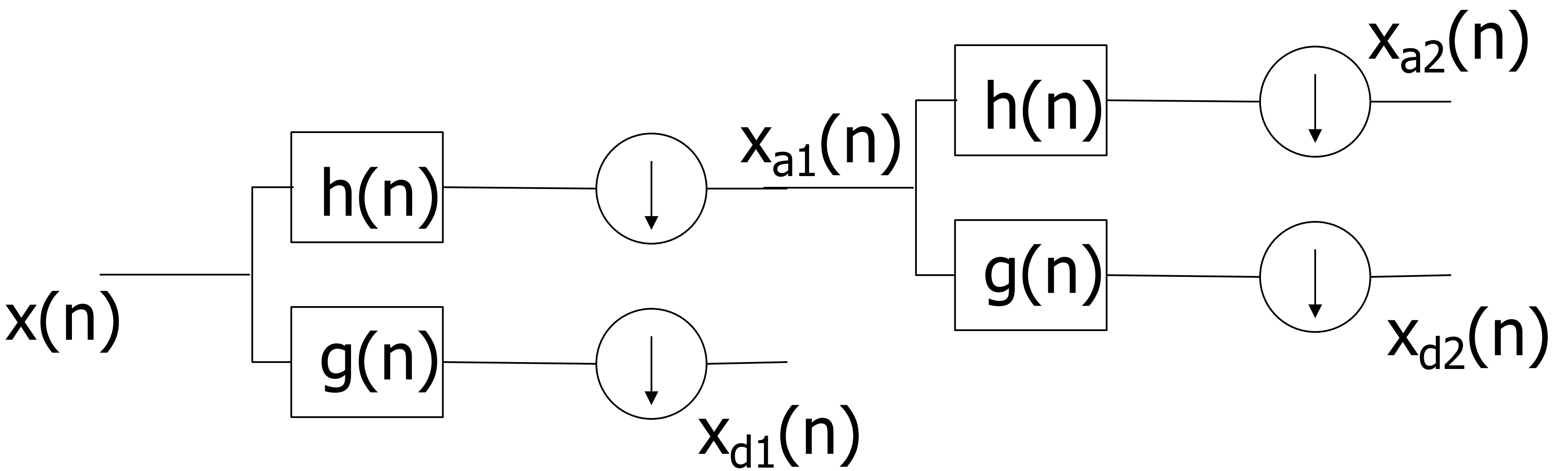
Courtesy: “Image and video processing in the compressed domain”, J. Mukhopadhyay, CRC Press, 2011.

DWT

- Translated only at discrete grid points.
- Scaled by powers of 2: 2^j, $j=0,1,...$
  - Downsampling takes care of dilation of wavelets and allows to use the same function at that level.
  - Filtering by the filter of same impulse response.
- Filtering banks.



Dyadic decomposition



- At each level sample size is halved
  - Equivalent of scaling by 2.
- Total number of samples remain the same.

Typical wavelet filters

- Daubechies 9/7 filters

n	Analysis filter bank		Synthesis filter bank	
	$h(n)$	$g(n-1)$	$h'(n)$	$g'(n+1)$
0	0.603	1.115	1.115	0.603
± 1	0.267	-0.591	0.591	-0.267
± 2	-0.078	-0.058	-0.058	-0.078
± 3	-0.017	0.091	-0.091	0.017
± 4	0.027			0.027

- Le Gall 5/3 filters

n	Analysis filter bank		Synthesis filter bank	
	$h(n)$	$g(n-1)$	$h'(n)$	$g'(n+1)$
0	$\frac{6}{8}$	1	1	$\frac{6}{8}$
± 1	$\frac{2}{8}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2}{8}$
± 2	$-\frac{1}{8}$			$-\frac{1}{8}$

Courtesy: "Image and video processing in the compressed domain", J. Mukhopadhyay, CRC Press, 2011.

2-D DWT

- Separable filters.
- Transform rows, then transform columns.

By 5/3 Analysis filters

Applications:

- Compression
- Denoising
- Feature representation
- Image fusion

…

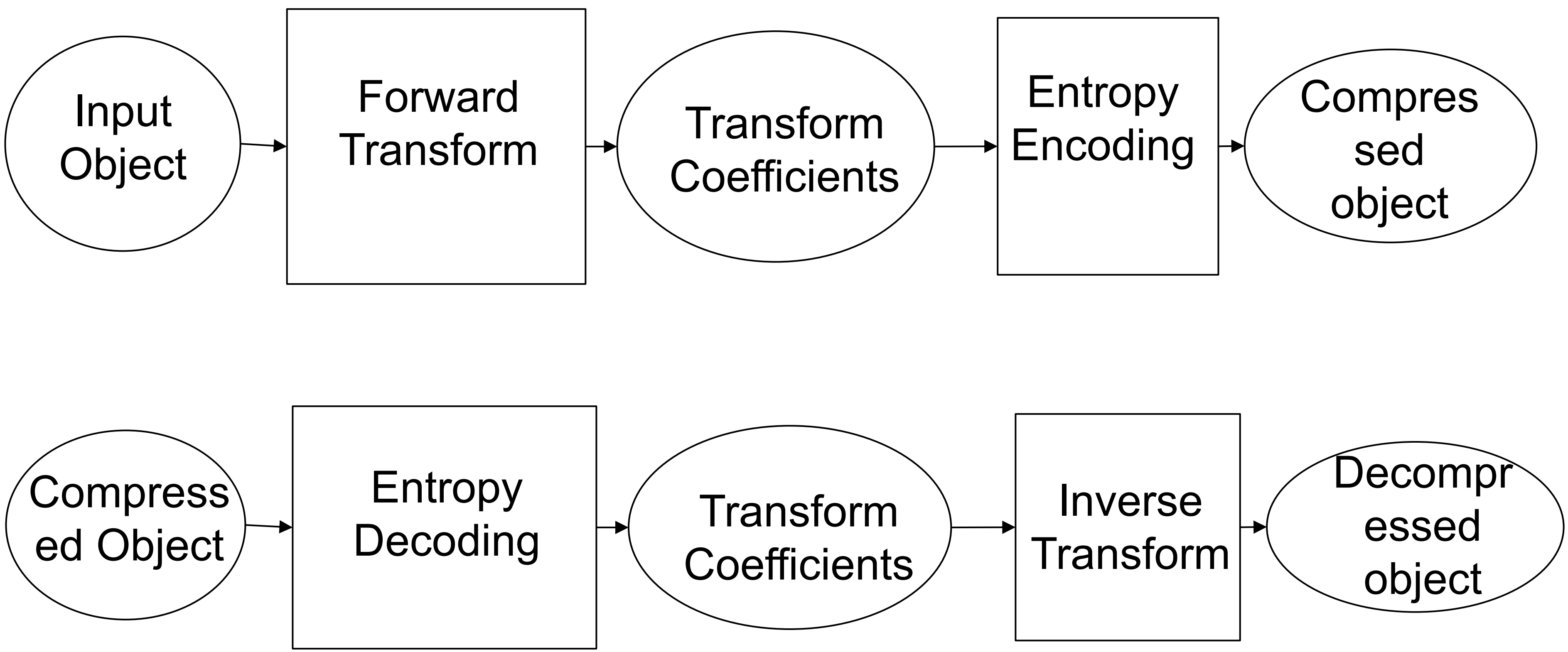
Image compression

- An alternative representation requiring less storage compared to in original original space.
  - An analogy with representation of a circle:
    - A set of all points in its periphery.
    - Only three (non-collinear) points.
    - Center and radius
- Decompression: Reconstruction from a compressed image in the original space.
- Lossy compression: Approximate reconstruction.
- Lossless compression: Exact reconstruction

Desirable features

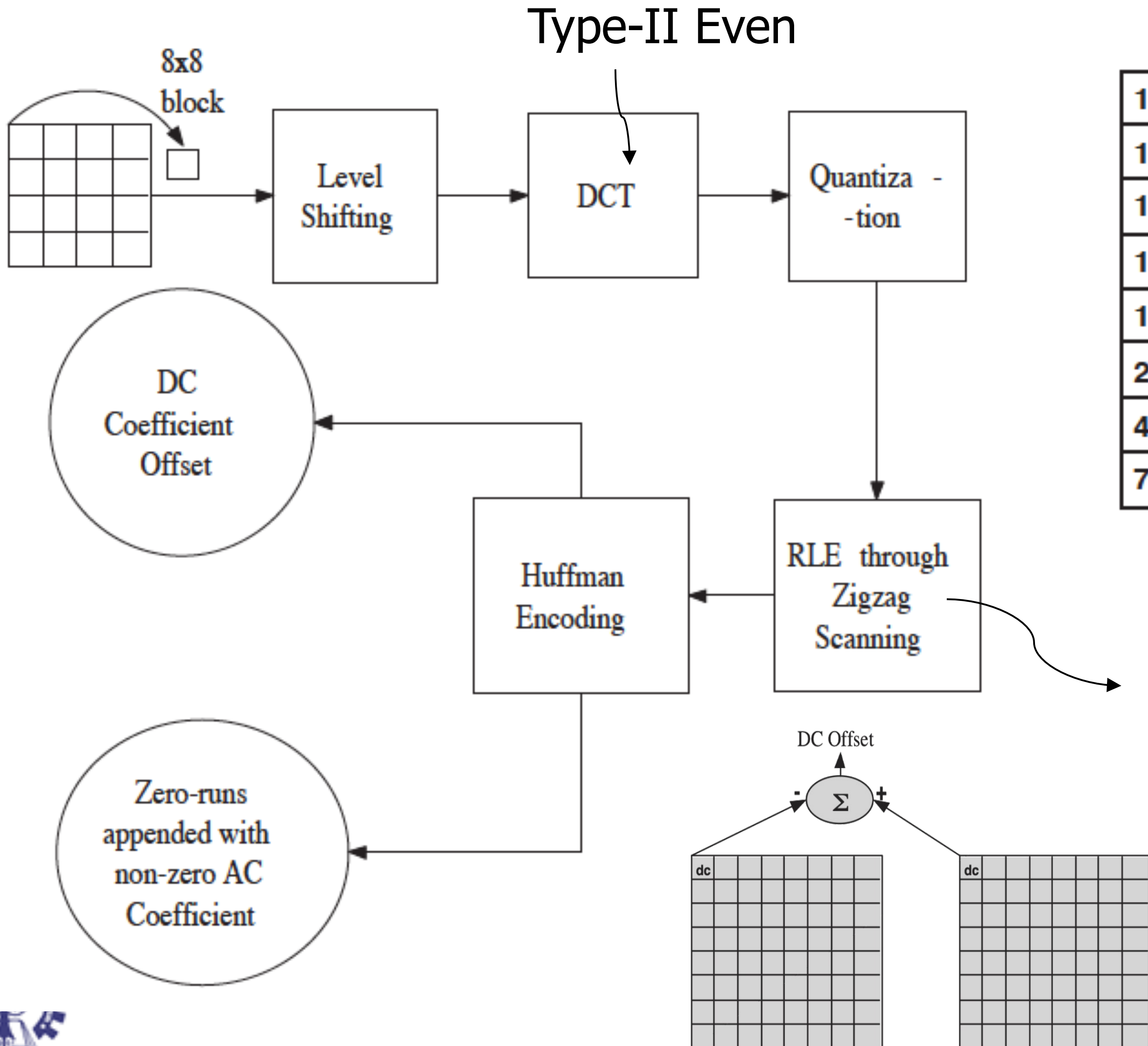
- Good reconstructibility
  - Visual quality of decompressed image should be high.
- Low redundancy
  - Spatial correlation, Channel (color) correlation, Symbol representation,
- Factorization in substructures
  - Frequency components, Space-frequency decomposition,
    - DCT, DWT

Generic pipeline of compression and decompression



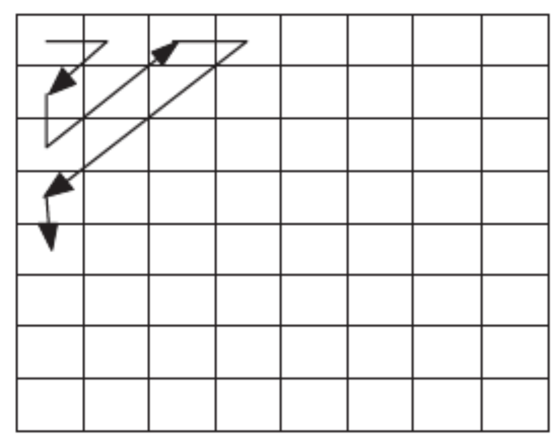
Courtesy: J. Mukhopadhyay, Image and video processing in the compressed domain, CRC Press, 2011.

JPEG: Baseline scheme



Typical quantization mask.

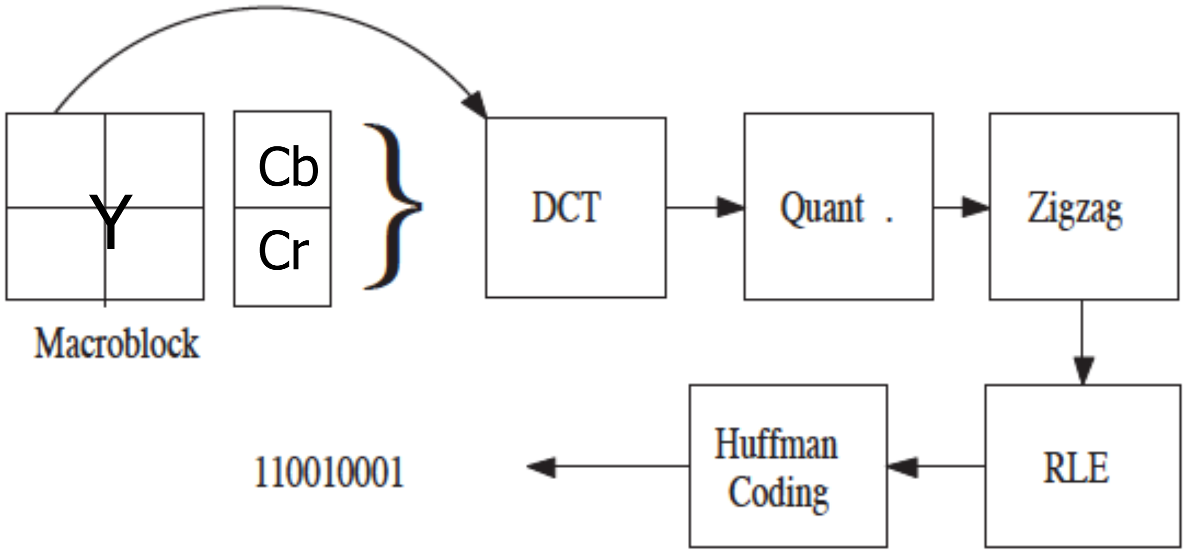
16	11	10	16	24	40	51	61
12	12	14	19	26	58	60	55
14	13	16	24	40	57	69	56
14	17	22	29	51	87	80	62
18	22	37	56	68	109	103	77
24	35	55	64	81	104	113	92
49	64	78	87	103	121	120	101
72	92	95	98	112	100	103	99



Color encoding in JPEG

- Y-Cb-Cr color space:

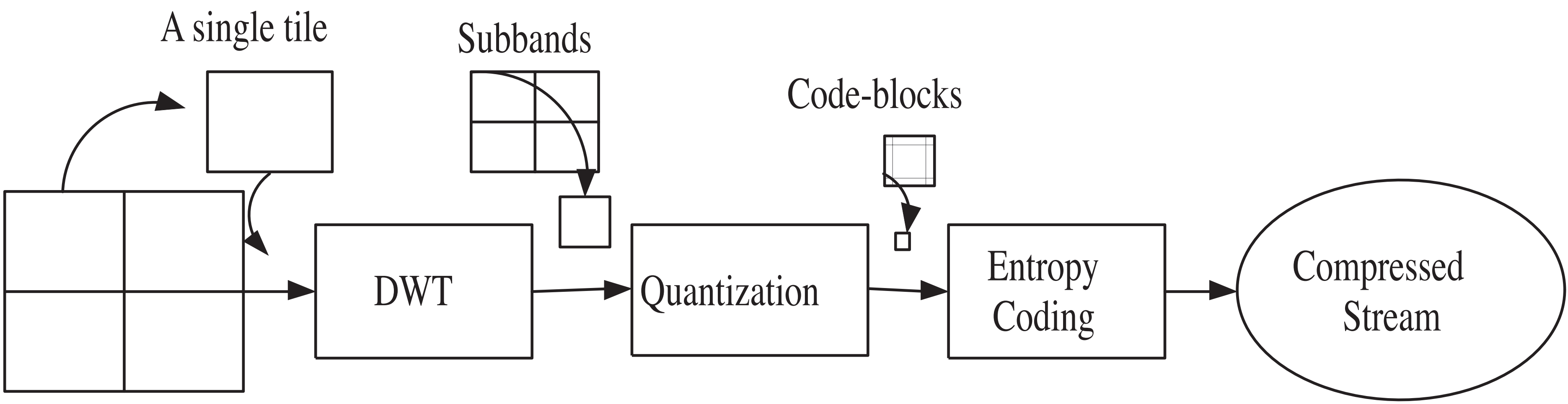
$$Y \;= 0.520G \;+ 0.098\,B + 0.256R$$
$$Cb = -0.290G + 0.438\,B - 0.148R + 128$$
$$Cr = -0.366G \;- 0.071\,B + 0.438R + 128$$



Courtesy: J. Mukhopadhyay, Image and video processing in the compressed domain, CRC Press, 2011.

Courtesy: J. Mukhopadhyay, Image and video processing in the compressed domain, CRC Press, 2011.

JPEG 2000



- Lossy: Daubechies 9/7 filters
- Lossless: Le Gall 5/3 filters
- Color Transformation:
  - Lossy: Y-Cb-Cr (w/o downsampling)
  - Lossless: $Y = \left\lfloor \frac{R+2G+B}{4} \right\rfloor$ $U=R-G$ $V=B-G$

$$G = Y - \left\lfloor \frac{U+V}{4} \right\rfloor$$

$$R=U+G$$

$$B=V+G$$

JPEG 2000: Quantization

- Each sub-band independently quantized with a uniform quantization threshold.

$$\Delta = 2^{n-\epsilon}(1 + \frac{\mu}{2^{11}})$$

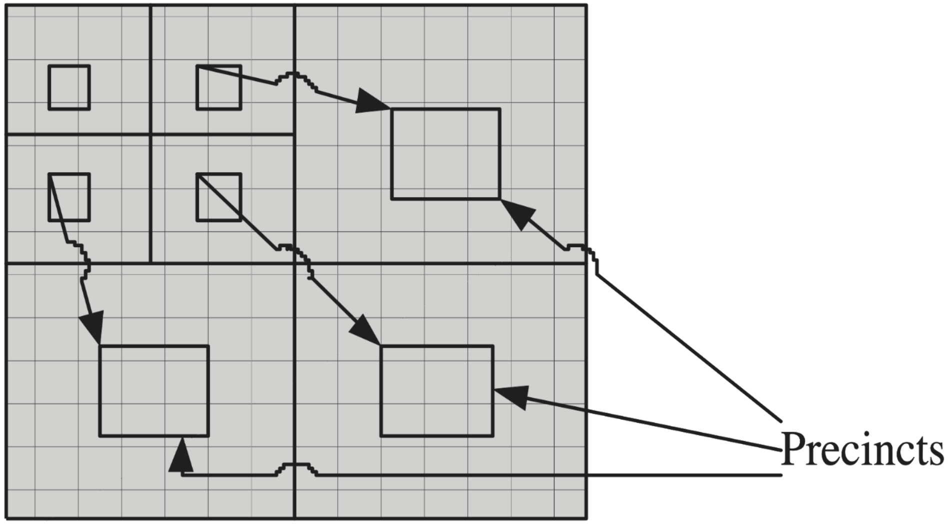
  - n: Nominal dynamic range of the sub-band, e.g. 10 for HH_1
  - ϵ , μ : the number of bits allotted to the exponent and mantissa respectively, of its coefficients.
- Quantized coefficient (of $X(u,v)$)

$$X_q(u,v) = sign(X(u,v)) \left\lfloor \frac{|X(u,v)|}{\Delta} \right\rfloor$$

- For lossless compression: $\mathbf{\Delta}=1$
- Implicit quantization: Lowest level (LL): ϵ_0 , μ_0
- For the i th sub-band at level k: $\mu_i = \mu_0$ and $\epsilon_i = \epsilon_0 + i - k$

JPEG2000: Code Structure

- Every sub-band partitioned into a set of non-overlapping codeblocks.
- Each codeblock independently coded by a schema called Embedded Block Coding on Truncation (EBCOT).
- each bit-plane of wavelet coefficients is processed by three passes, namely, significant propagation, magnitude refinement, and clean-up.
- The resulting bit-stream encoded using Arithmetic Encoding.
- A layer formed with the output of similar passes from a group of code blocks.
- In a layer, packets formed by grouping corresponding code blocks of subbands at the same level of decomposition.
  - also known as precincts



Courtesy: J. Mukhopadhyay, Image and video processing in the compressed domain, CRC Press, 2011.

Summary

- Image transforms involve representation of images as a linear combination of a given set of basis functions.
- For a finite discrete sequence, this is treated as a linear combination of a given set of basis vectors.
- Orthogonal set of basis functions (vectors) simplifies computation of forward and inverse transforms
  - Inner product of the function with the basis function.
  - Examples: Fourier Transform, Wavelet Transforms (may be also non-orthogonal)
- A set of basis functions may be orthogonal but not complete for exactly representing any arbitrary function.
  - Cosine and Sine Transforms in continuous domain.
  - For finite discrete sequences several orthogonal and complete transforms available: DFT, GDFTs, DCTs, DSTs, etc.

Summary

- Alternative representation provides other insights of structure of images.
  - low frequency and high frequency components.
- May become useful for providing more compact representation.
  - A few transform coefficients.
  - Selective quantization of components, considering their effect on our perception.
    - Image compression: JPEG
- Sometimes convenient for processing.
  - Filtering, enhancement, ….

Summary

- Wavelets represent the scale of features in an image, as well as their positions.
  - Time-scale, Space-Scale representation
- Fast computation of forward and inverse transform
- Provides multiresolution representation.
  - Enables progressive and scalable processing
- Lossy and lossless reconstruction possible.
- useful for a number of applications including image compression.
  - JPEG2000

Thank You