Image Restoration

Jayanta Mukhopadhyay
Dept. of CSE,
IIT Kharagpur

Image smoothing / enhancement: Gaussian Filtering

- Low pass filtering
  - Gaussian filter
    - Freq. / Spatial domain

$$H(u,v) = Ae^{-\frac{4\pi^2(u^2+v^2)\sigma^2}{2}}$$

$$\Updownarrow$$

$$h(m,n) = \frac{A}{4\pi\sigma^2}e^{-\frac{(m^2+n^2)}{2\sigma^2}}$$

- More complex filters
  - Using difference of Gaussian
  - LPF, BPF, HPF

$$H(u,v) = A_1e^{-\frac{4\pi^2(u^2+v^2){\sigma_1}^2}{2}} - A_2e^{-\frac{4\pi^2(u^2+v^2){\sigma_2}^2}{2}}$$

$$\Updownarrow$$

  - DoG in spatial domain, too!

$$h(m,n) = \frac{A_1}{4\pi{\sigma_1}^2}e^{-\frac{(m^2+n^2)}{2\sigma1^2}} - \frac{A_2}{4\pi{\sigma_2}^2}e^{-\frac{(m^2+n^2)}{2\sigma2^2}}$$

Filtering in 1-D the transform domain

- Freq. Shifting $DFT\left(f(n)e^{j2\pi\frac{k_0}{N}n}\right) = F(<k-k_0>_N)$
- Phase Shifting $IDFT\left(F(k)e^{-j2\pi\frac{k}{N}n_0}\right) = f(<n-n_0>_N)$

$$F(k) \longrightarrow \boxed{H(k)} \longrightarrow G(k)=H(k)\,F(k)$$

- Use sufficient 0 padding at the both end to make circular convolution equivalent to linear convolution
  - To take care of boundary effect.
  - The length of $f(n)$ and $h(n)$ should be the same.
- $H(k)$ usually provided as symmetric about the center.
  - 0^{th} freq. at the N/2 th element.
- Center $F(k)$ as $F_c(k)$ by multiplying f(n) with $(-1)^n$ ← ($k_0=N/2$)
- Obtain $G(k)= H(k)\ .\ F_c(k)$
- Multiply $G(k)$ by $(-1)^k$ and perform IDFT to get $g(n)$.← ($n_0=N/2$)

Low pass filters

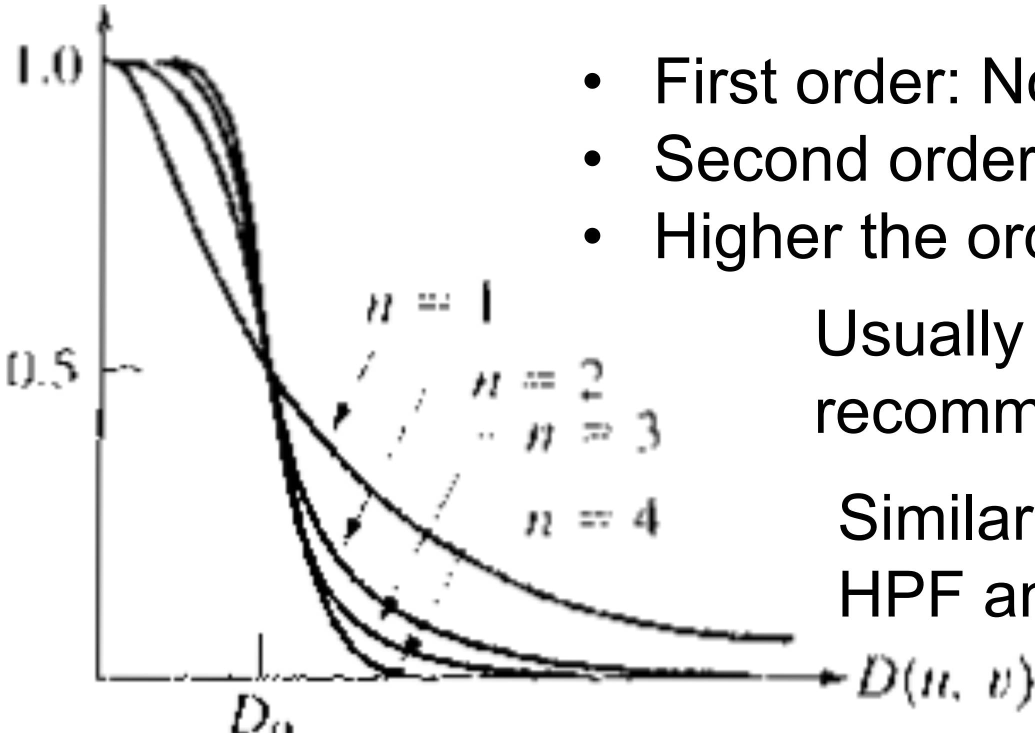
- Ideal LPF
  - $H(u,v) = 1$, if $D(u,v) \le D_0$

    $= 0$, Otherwise
- Use centered freq. transform:
  - $D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$
- Make DFT of image also centered
- Perform filtering
- Not very practical
  - Blurring and ringing effect prominent!
  - Sharp discontinuity in frequency response!
  - Impulse response in the form of a Sinc function.

Butterworth Low Pass Filters

- Transfer function of order n:
  - Avoids sharp discontinuity
  - 50% of maximum as $D(u,v)=D_0$

$$H(u,v) = \frac{1}{1 + \left(\frac{D(u,v)}{D_0}\right)^{2n}}$$



- First order: No ringing
- Second order: Ringing imperceptible
- Higher the order ringing effect higher.

Usually BPLF of order 2 recommended.

Similarly we may have HPF and BPF.

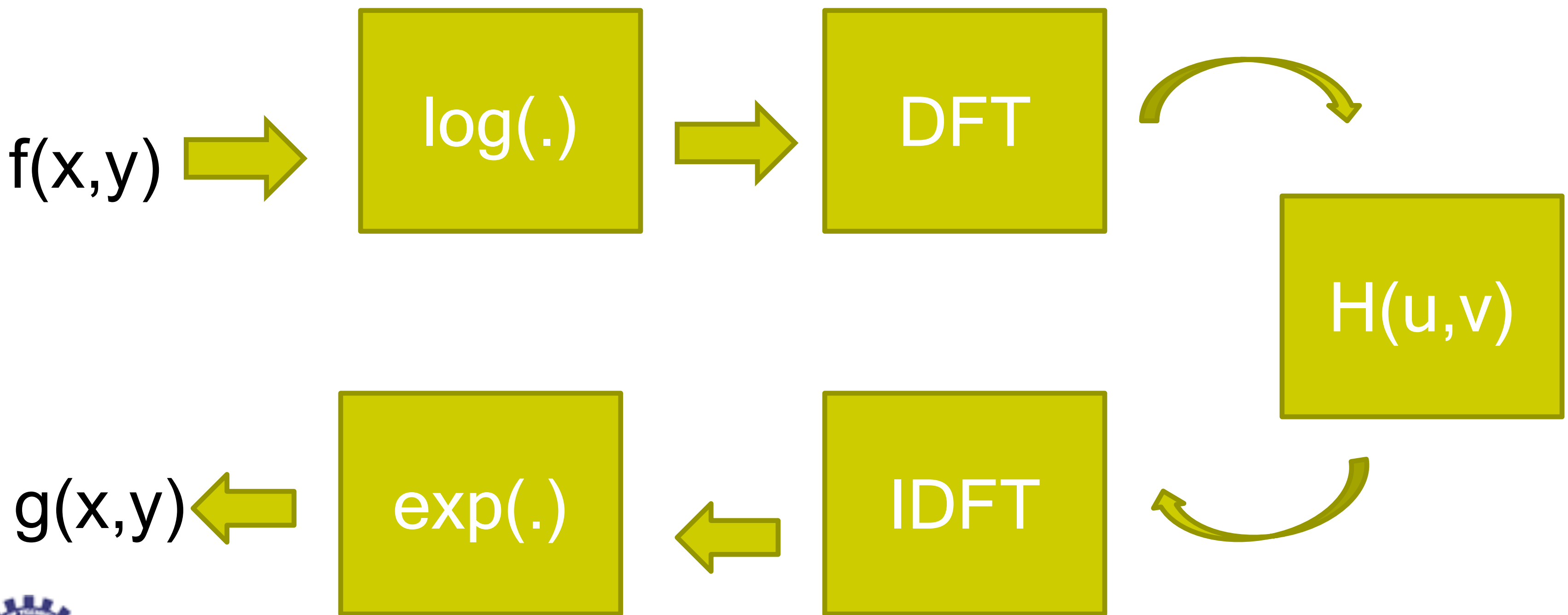
Homomorphic filtering

- Let $f(x,y)=i(x,y)\,r(x,y)$
  - $i(x,y)$: Illumination variation
    - Slow / Low frequency
  - $r(x,y)$: reflectance variation
    - High / High frequency
- Make additive component in the log domain
- Apply filtering by suppressing low frequencies and enhancing high frequencies.
  - Dynamic range compression coupled with contrast enhancement
- Back to the original domain by exponentiation.

Homomorphic filtering

- A typical example
  - γ_L<1, γ_H>1

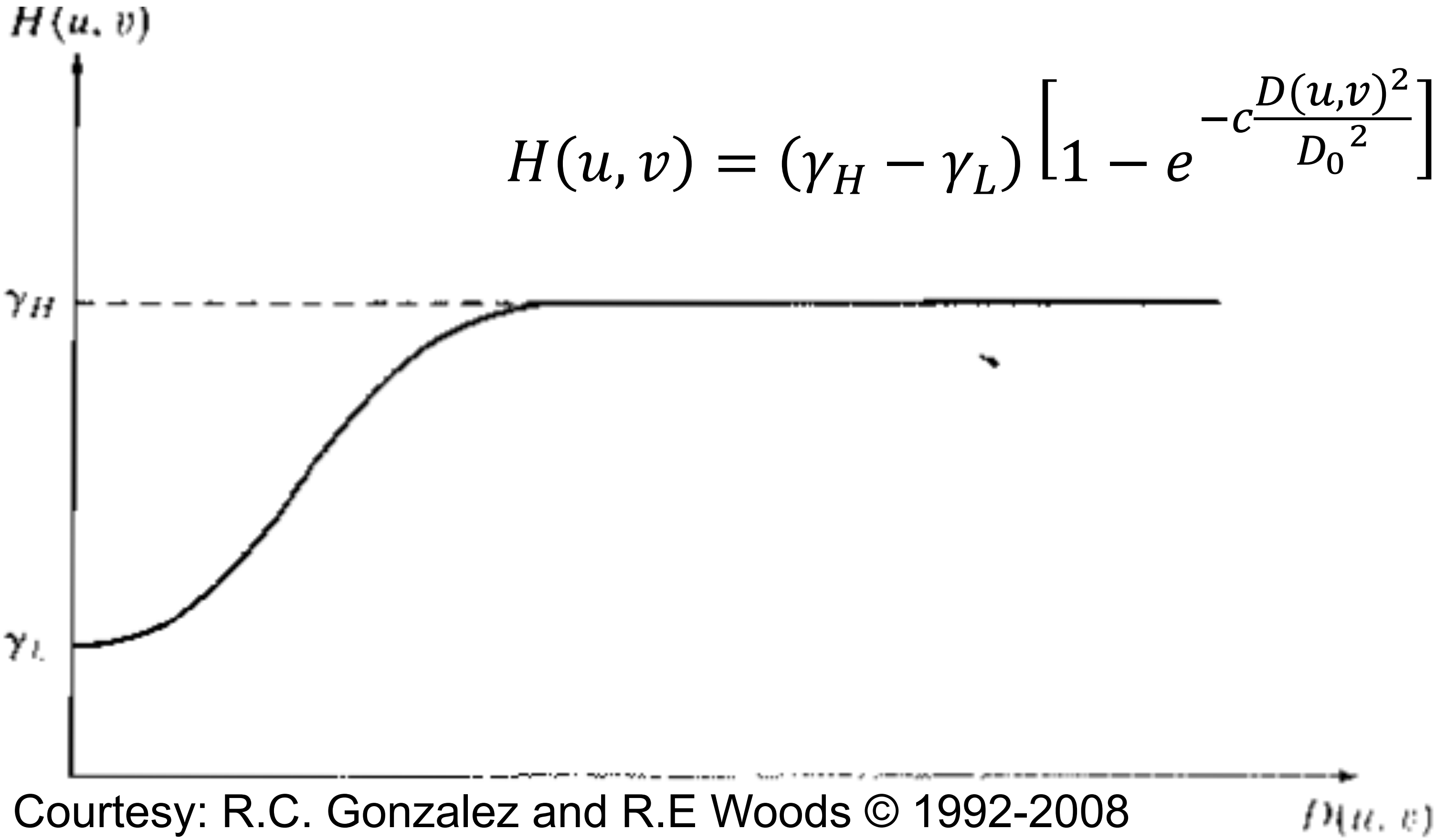
$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\frac{D(u,v)^2}{{D_0}^2}}\right] + \gamma_L$$



Homomorphic filtering: Typical example

- Simultaneous dynamic range compression and contrast enhancement

$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\frac{D(u,v)^2}{{D_0}^2}}\right] + \gamma_L$$



Courtesy: R.C. Gonzalez and R.E Woods © 1992-2008

Band reject filters (BRF)

BPF=1-BRF

Distance from the center

- Ideal

$$H(u,v) = \begin{cases} 0 & -\frac{W}{2} + D_0 \leq D(u,v) \leq D_0 + \frac{w}{2} \\ 1 & Otherwise \end{cases}$$

Cut-off frequency

- Butterworth

$$H(u,v) = \frac{1}{1 + \left[\frac{W.D(u,v)}{D^2(u,v) - D_0^2}\right]^{2n}}$$

- Gaussian

$$H(u,v) = 1 - e^{-\left[\frac{D^2(u,v) - D_0^2}{wD(u,v)}\right]^2}$$

Used for removing periodic noise.

Notch filters

- A notch filter rejects or passes frequencies in a predefined frequency about the center of the frequency rectangle.
- Zero phase shift filter should be symmetric about the origin.
  - If there exists a notch at center (u_0,v_0), there must be a notch at $(-u_0,-v_0)$.
  - A general form:

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v)\, H_{-k}(u,v)$$

HPF with center at (u_k,v_k) or (u_{-k},v_{-k})

  - e.g. using Butterworth HPFs

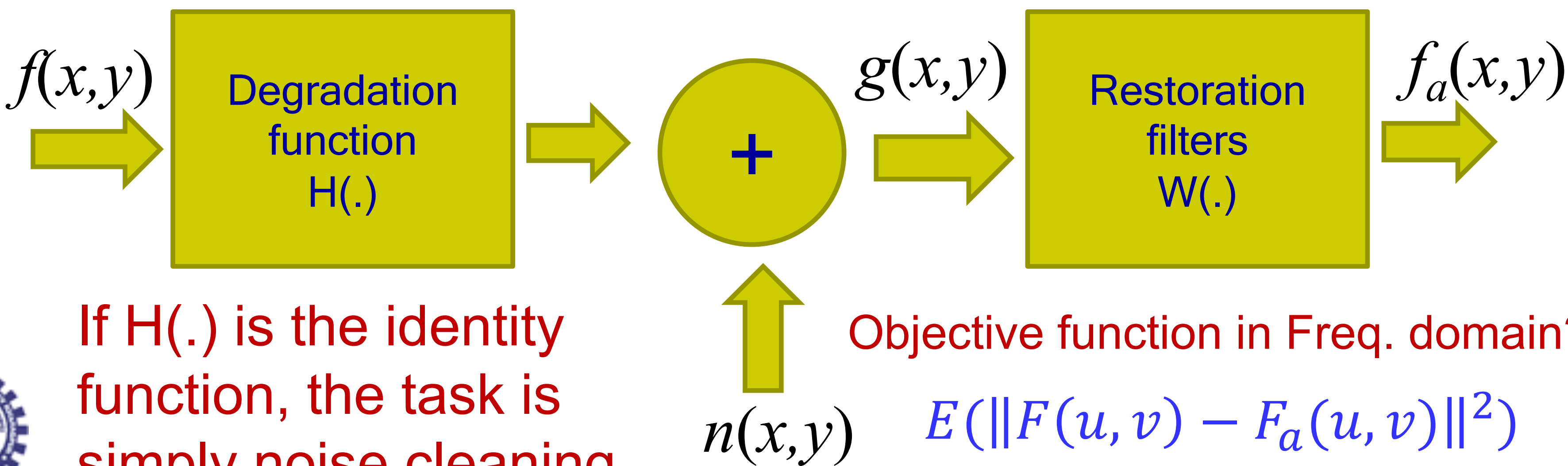
$$H_k(u,v) = \frac{1}{1+\left[\frac{D_{0k}}{D_k(u,v)}\right]^{2n}}$$

$$D_k(u,v) = \left[\left(u-\frac{M}{2}-u_k\right)^2 + \left(v-\frac{N}{2}-v_k\right)^2\right]^{\frac{1}{2}}$$

A degradation model and restoration

- Degradation function:
  - Identity (No degradation), Linear filter (Motion Blur), Transformation of a functional value
- Noise model
  - White, Gaussian, Rayleigh,

- Linear degradation model.
  $g(x,y) = h(x,y) * f(x,y) + n(x,y)$
- To design a restoration filter w(x,y)
  - such that w(x,y)*g(x,y) = f_a(x,y) close to f(x,y).
- To minimize
  $E(\|f(x,y) - f_a(x,y)\|^2)$



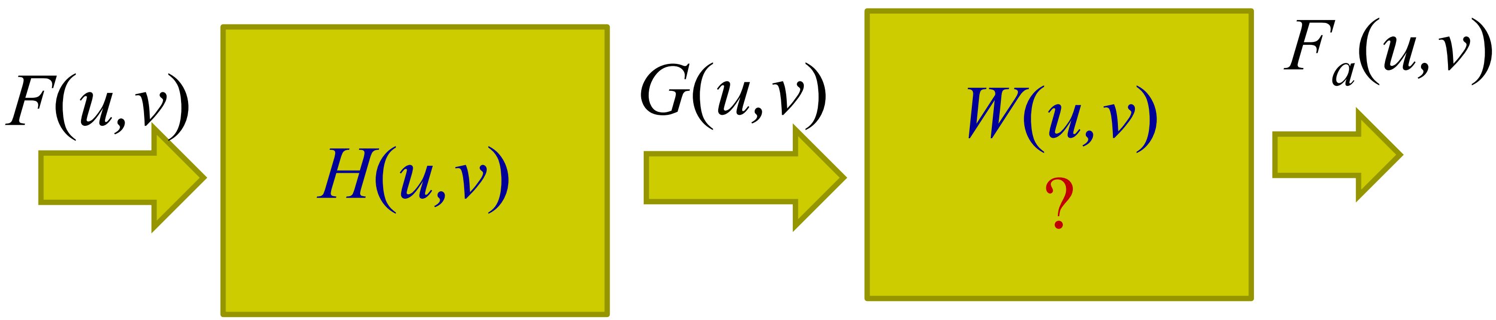
If H(.) is the identity function, the task is simply noise cleaning.

Objective function in Freq. domain?

$E(\|F(u,v) - F_a(u,v)\|^2)$

Restoration in the absence of noise

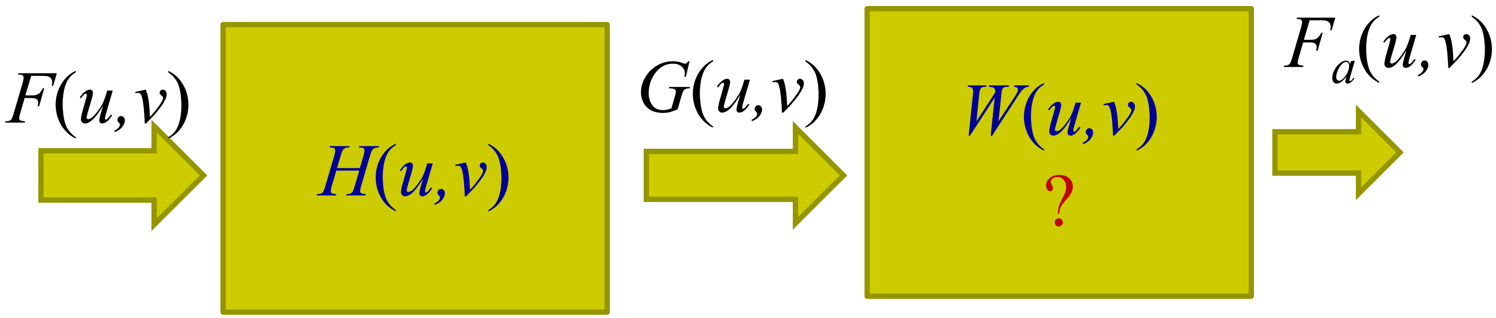
$$g(x,y) = h(x,y) * f(x,y) \quad \Leftrightarrow \quad G(u,v)=H(u,v)\,F(u,v)$$



- Inverse filtering: $W(u,v) = 1/H(u,v)$
  - Problem with zeros and low values in $H(u,v)$
- Minimize $E(||F(u,v) - F_a(u,v)\ ||^2)$

$$= E(||F(u,v) - W(u,v)\ H(u,v)\ F(u,v)\ ||^2)$$

Restoration in the absence of noise



- Power Spectrum of the image: $S_f(u,v)=||F(u,v)||^2=F^*(u,v)F(u,v)$
- To minimize $E_w=E(||F(u,v) - W(u,v)\, H(u,v)\, F(u,v)\, ||^2)$

For convenience, F(u,v) written as F and the same for all others.

$$E_w=||F||^2-(W^*H^*+WH)||F||^2+||W||^2||H||^2||F||^2$$

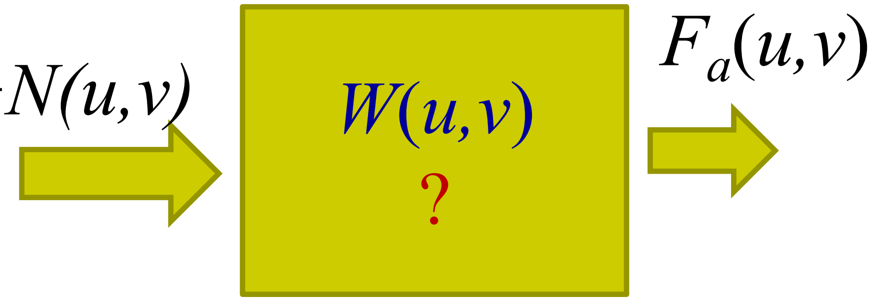
$$\frac{\partial E_w}{\partial W(u,v)}=0 \Rightarrow -HS_f+W^*||H||^2S_f=0 \Rightarrow W=H^*/||H||^2=1/H$$

Pretending W^* constant for W

Inverse filtering!
The same problem!

Restoration in the presence of noise

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$



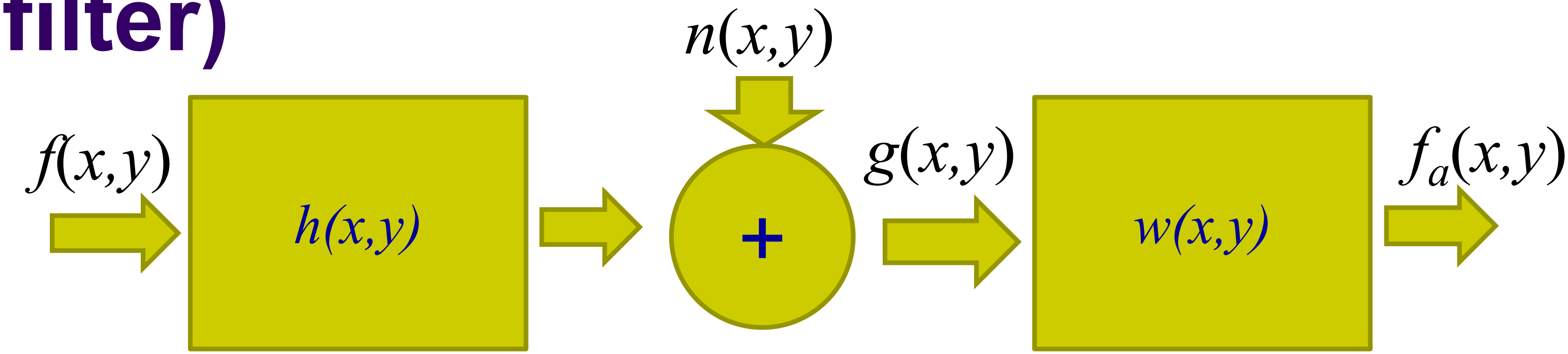
- Power Spectrum of the original image: $S_f = ||F||^2 = F^*F$
- Power Spectrum of the noisy image: $S_g = ||G||^2 = G^*G$
  - $S_g = ||H||^2\,||F||^2 + H^*F^*N + HFN^* + ||N||^2$
- As noise assumes to be uncorrelated: $E(F^*N) = E(N^*F) = 0$
- Hence, $S_g = ||H||^2\,||F||^2 + ||N||^2 = ||H||^2 S_f + S_n$
- To minimize $E_w = E(||F - WG||^2)$

$$W = (H^*S_f) / (||H||^2 S_f + S_n)$$

  - $E_w = E(||F||^2 - FW^*G^* - WGF^* + ||W||^2\,||G||^2)$
  - $= E(||F||^2 - FW^*G^* - WHF^*F - WF^*N + ||W||^2\,||G||^2)$

$$\frac{\partial E_w}{\partial W(u,v)} = 0 \Rightarrow -HS_f + W^*(||H||^2 S_f + S_n) = 0$$

Weiner Filter (Least square error filter)



- Solution in frequency domain: $W=(H^*S_f)/\,(||H||^2 S_f+S_n)$

$$W = \frac{H^*}{||H||^2 + \frac{S_n}{S_f}}$$

$$W = \frac{H^*}{||H||^2 + K}$$

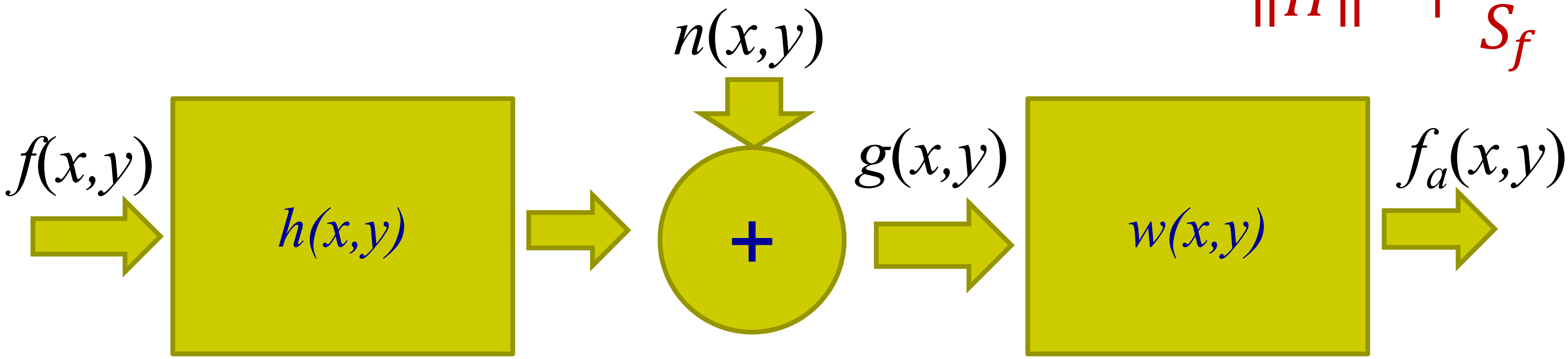
Noise to Signal Ratio

$$W = \frac{1}{H}\frac{||H||^2}{||H||^2 + K}$$

Weighted Inverse filter!

Tasks of restoration

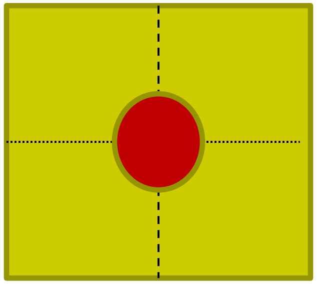
$$W = \frac{H^*}{\|H\|^2 + \frac{S_n}{S_f}}$$



- Model degradation:
  - Design / derive $h(x,y)$
- Model Noise
  - Identify PDF and estimate parameters
- Derive W or $w(x,y)$
- Apply filtering: $f_a(x,y) = w(x,y) * g(x,y)$

Degradation of a defocused image

- Defocused image: The projected point is not sharp.
- e.g. The projection forms a circle of radius r.
  - Spatial resolution along x: Δx and y: Δy



N: Total number of pixels within the circle

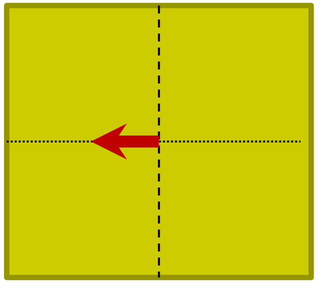
h(i,j)

$$h(i\,,\,j) = 1/N,\ (i.\ \Delta x\ )^2 + (j.\ \Delta y\ )^2 \leq r$$
$$= 0 \qquad \text{Otherwise}$$

Degradation due to motion blur

- Motion blur: Movement of camera, Movement of object.
  - e.g. The camera moving with a velocity v_x in the direction of x
    - Exposure time: t
    - Spatial resolution along x: Δx
    - Number of pixels covered in a shot due to movement: $N=(v_x t) / \Delta x$

$$h(i,j)= 1/N, \quad -(N-1) \leq i \leq 0$$
$$= 0 \quad \text{Otheriwse}$$



h(I,j)

Degradation model for a general planar rigid body motion

- Time varying translational motion vector : $x_0(t)\, i + y_0(t)\, j$
- Duration of exposure: T
  - Shutter opening and closing takes place instantaneously
- Blurred image $g(x,y)$ given by:

$$g(x,y) = \int_0^T f(x - x_0(t), y - y_0(t))\, dt$$

Fourier transform of $g(x,y)$:

$$G(u,v) = \int_0^T F(u,v) e^{-j2\pi(ux_0(t)+vy_0(t))}\, dt$$

Degradation model for a general planar rigid body motion

$$g(x,y) = \int_0^T f(x - x_0(t), y - y_0(t))\, dt$$

$$G(u,v) = \int_0^T F(u,v) e^{-j2\pi(ux_0(t)+vy_0(t))}\, dt$$

$$G(u,v) = F(u,v)\underbrace{\left[\int_0^T e^{-j2\pi\left(ux_0(t)+vy_0(t)\right)}\, dt\right]}_{H(u,v)}$$

Motion blur special cases

- Uniform linear motion in horizontal direction:
  - $x_0(t)=at/T$, and $y_0(t)=0$

$$H(u,v)=\int_0^T e^{-j2\pi\left(ux_0(t)+vy_0(t)\right)}\,dt$$

$$H(u,v)=\int_0^T e^{-j2\pi\left(\frac{uat}{T}\right)}\,dt$$

$$=\frac{T}{\pi ua}\sin(\pi ua)e^{-j\pi ua}$$

Motion blur special cases

- Uniform linear motion in a plane:
  - $x_0(t)=at/T$, and $y_0(t)= bt/T$

$$H(u,v) = \int_0^T e^{-j2\pi\left(ux_0(t)+vy_0(t)\right)}\, dt$$

$$H(u,v) = \int_0^T e^{-j2\pi\left(\frac{uat}{T}+\frac{vbt}{T}\right)}\, dt$$

$$= \frac{T}{\pi(ua+vb)} \sin(\pi(ua+vb)) e^{-j\pi(ua+vb)}$$

Atmospheric degradation model (Hufnagel & Stanley (1964))

$$H(u,v) = e^{-k(u^2+v^2)^{\frac{5}{6}}}$$

- k is a constant that depends on the turbulence.
- Almost the form of a Gaussian function
  - LPF

How do you relate them in discrete domain?

Use normalized frequency: k/N for sequence of length N.

Variance is the measure of noise power.

White noise with a pdf has the same power at every freq.

Noise models

- Gaussian:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$mean = \mu \quad var = \sigma^2$$

- Rayleigh:

$$p(x) = \begin{cases} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & for \quad x \geq a \\ 0 & for \quad x < a \end{cases}$$

$$mean = a + \sqrt{\frac{\pi b}{4}}$$

$$var = \frac{b(4-\pi)}{4}$$

- Erlang (Gamma):

$$p(x) = \begin{cases} \frac{a^b x^{b-1}}{(b-1)!} e^{-ax} & for\ x \geq 0 \\ 0 & for\ x < a \end{cases}$$

$$mean = \frac{b}{a} \quad var = \frac{b}{a^2}$$

Noise models

- Exponential:

$$mean = \frac{1}{a} \quad var = \frac{1}{a^2}$$

$$p(x) = \begin{cases} ae^{-ax} & for\ x \geq 0 \\ 0 & for\ x < 0 \end{cases}$$

- Uniform:

$$mean = \frac{a+b}{2}$$

$$var = \frac{(b-a)^2}{12}$$

$$p(x) = \begin{cases} \frac{1}{b-a} & for\ a \leq x \leq b \\ 0 & Otherwise \end{cases}$$

- Impulse (Salt and Pepper):

Bipolar impulse

$$p(x) = \begin{cases} P_a & for\ x = a \\ P_b & for\ x = b \\ 0 & Otherwise \end{cases}$$

Estimation of noise

- Study relatively flat (constant) region and study the histogram.
- The shape of histogram may indicate appropriate PDF to be chosen.
- Compute mean and variance of the flat region.
- Relate to them to the parameters of the distribution. (Optional)
- Make robust estimation of variance (by sampling and estimation using the central limit theorem)
- Use variance for the noise power at every frequency (modeling as white noise)

Noise removal: linear and nonlinear filters exploiting local statistics

- Arithmetic mean. $\frac{1}{N}\sum_{i=0}^{N-1} x_i$
- Geometric mean. $\left(\prod_{i=0}^{N-1} x_i\right)^{\frac{1}{N}}$
- Harmonic mean. $\dfrac{1}{\frac{1}{N}\sum_{i=0}^{N-1}\frac{1}{x_i}}$
- Contraharmonic mean. $\dfrac{\sum x_i^{Q+1}}{\sum x_i^{Q}}$

  Q: Order of filter

  Q= 0 (A.M.), Q= -1 (H.M.)

- Order Statistics.
  - Median
  - Max
  - Min
  - Mid-point
    - Average of Max and Min.
  - Alpha-trimmed mean
    - Mean excluding top (d/2) and bottom (d/2) in the rank order.

Adaptive filter for restoration

- Exploit local statistics
  - Local mean: $\boldsymbol{\mu}$ Local variance: $\boldsymbol{\sigma}^2$
  - Local noise variance: η^2
  - Pixel value: $g(x,y)$
- Desirable
  - If $\eta^2=0$ return $g(x,y)$
  - If $\boldsymbol{\sigma}^2$ high return close to $g(x,y)$
  - If $\eta^2 = \boldsymbol{\sigma}^2$ return local mean $\boldsymbol{\mu}$
- Adaptive expression

$$f_a(x,y) = g(x,y) - \frac{\eta^2}{\boldsymbol{\sigma}^2}(g(x,y) - \boldsymbol{\mu})$$

Constrained least squares filtering

$$W = \frac{H^*}{\|H\|^2 + \frac{S_n}{S_f}}$$

- Limitations of Wiener Filtering:
  - Power spectra of the undegraded image and noise required to be known.
  - Based on minimizing a statistical criterion.
    - Optimization in average sense
- Problem statement reformulated for taking care of individual image separately.
  - Criteria set from filtering in spatial domain
  - But solution obtained in the frequency domain

Constrained least squares filtering in Spatial domain

- Consider degraded and undegraded image in the form of a derasterized vector
  - $\mathbf{g}$ and $\mathbf{f}$ of length N, respectively.
- Let the degradation filter response given by a vector
  - $\mathbf{h}$ of length N
- The noise at each point expressed by a vector
  - η of length N
- The convolution operation can be expressed by a matrix operation as follows:
  - $\mathbf{g}=\mathbf{Hf}+\eta$
  - $\mathbf{H}$ is a matrix of dimension NxN

Optimization problem

- Minimize a smoothness criteria C (measured as sum of squares of Laplacians) subject to keeping noise at a desired level

$$C = \sum_{x=0}^{P-1} \sum_{y=0}^{Q-1} [\nabla^2 f(x,y)]^2$$

Subject to the constraint

A small positive value

$$\|\mathbf{g} - \mathbf{Hf}\|^2 = \|\boldsymbol{\eta}\|^2$$

or

$$abs(\|\mathbf{g} - \mathbf{Hf}\|^2 - \|\boldsymbol{\eta}\|^2) < a^2$$

Residual difference

Frequency domain solution

$$F(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |L(u,v)|^2} G(u,v)$$

- γ: a parameter, may be chosen iteratively such that the residual difference < a^2
- $L(u,v)$ is the Fourier Transform of the Laplacian operator $l(x,y)$

$$l(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Summary

- Filtering for noise removal
- Degradation model
- Weiner (LSE) filter
  - to model degradation filter and noise, and then obtain the LSE filter.
  - Apply Weiner filter on degraded image to restore it.
- Modeling motion blur and defocusing.
- Noise model
- Use of various local statistics for removing noise.
- Adaptive filter exploiting local statistics and variance of noise.
- Constrained least squares filters

Thank You