Public Key Cryptography

Euler's Theorem

- **a generalisation of Fermat's Theorem**
- **$M^{ø(n)} = 1 \pmod{n}$**
  - **for any M, n where gcd(M,n)=1**

- Corollary:

$$M^{1+k.ø(n)} = [(M^{ø(n)})^k \times M^1] \bmod n$$
$$= [(1)^k \times M] \bmod n$$
$$= M \bmod n$$

Public-Key Cryptography

- probably most significant advance in the 3000 year history of cryptography
- uses two keys – a public & a private key
- asymmetric since parties are not equal
- uses clever application of number theoretic concepts to function
- complements rather than replaces private key crypto

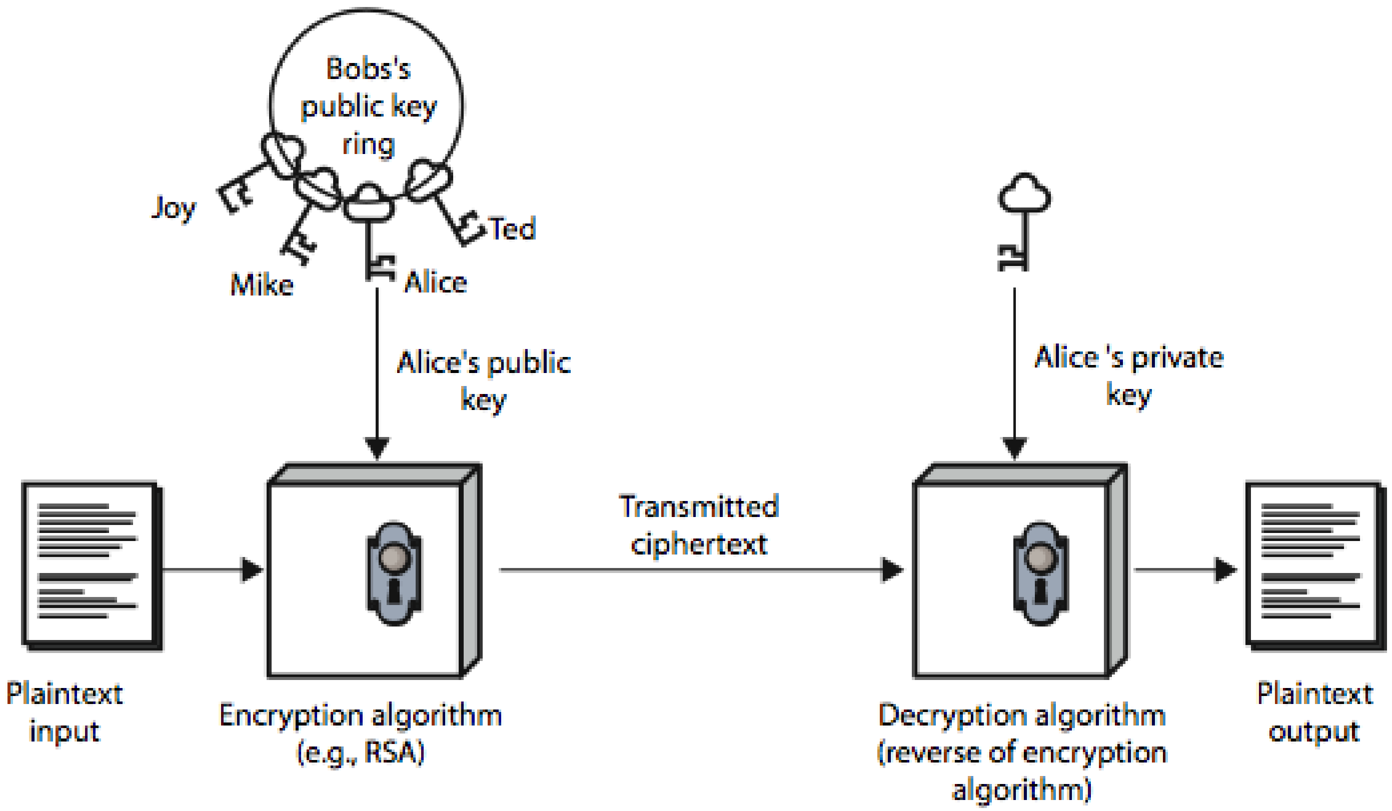
Why Public-Key Cryptography?

- developed to address two key issues:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- public invention due to Whitfield Diffie & Martin Hellman at Stanford Uni in 1976
  - known earlier in classified community

Public-Key Cryptography

- **public-key/two-key/asymmetric** cryptography involves the use of **two** keys:
  - a **public-key**, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**
- is **asymmetric** because
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

Public-Key Cryptography



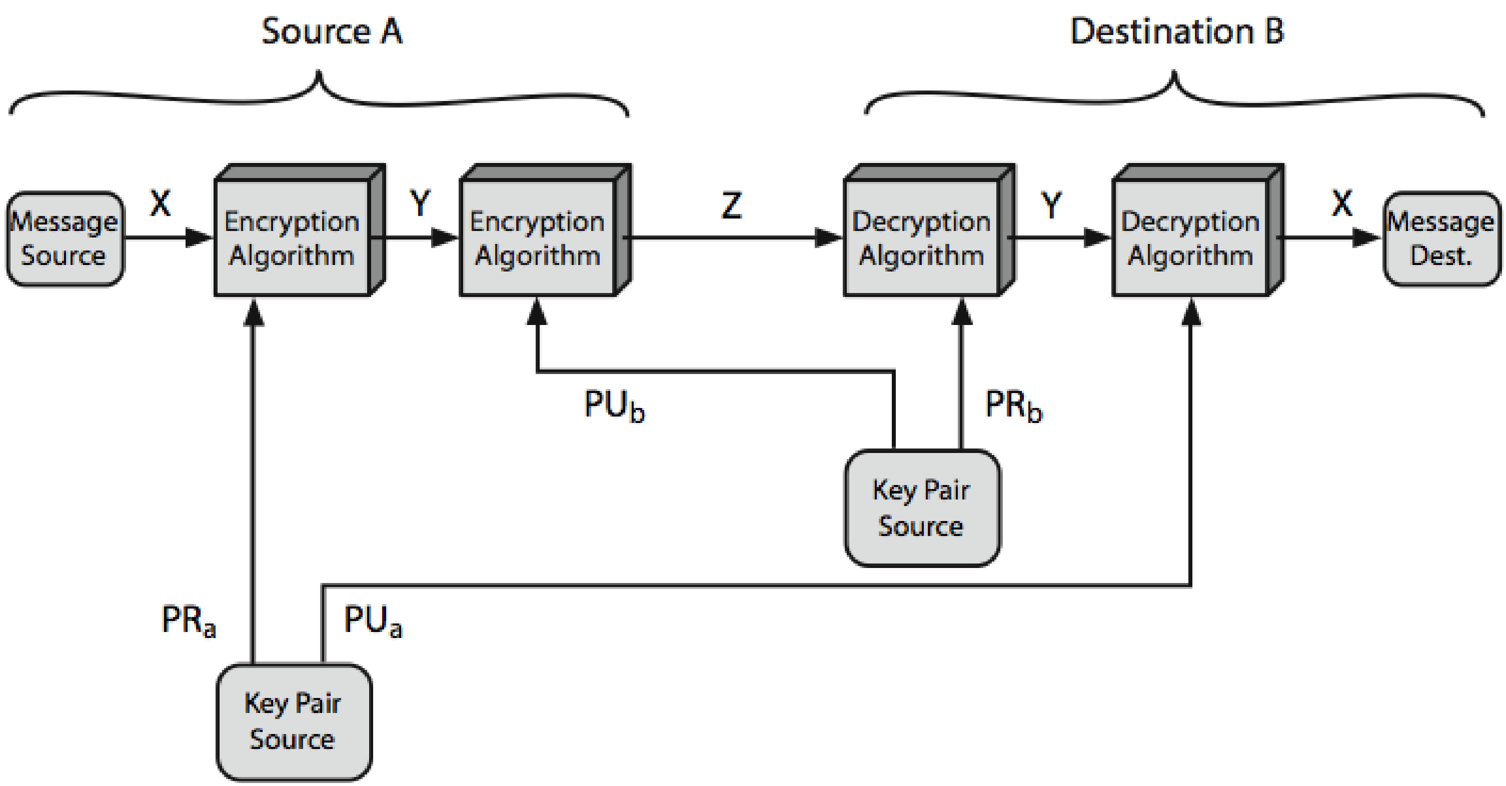
(a) Encryption

Stallings Book

Public-Key Characteristics

- Public-Key algorithms rely on two keys where:
  - it is computationally infeasible to find decryption key knowing only algorithm & encryption key
  - it is computationally easy to en/decrypt messages when the relevant (en/decrypt) key is known
  - either of the two related keys can be used for encryption, with the other used for decryption (for some algorithms)

Public-Key Cryptosystems



Stallings Book

Public-Key Applications

- can classify uses into 3 categories:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Security of Public Key Schemes

- like private key schemes brute force **exhaustive search** attack is always theoretically possible
- but keys used are too large (>1024 bits)
- security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (cryptanalyse) problems
- more generally the **hard** problem is known, but is made hard enough to be impractical to break
- requires the use of **very large numbers**
- hence is **slow** compared to private key schemes

RSA

- by Rivest, Shamir & Adleman of MIT in 1977
- best known & widely used public-key scheme
- based on exponentiation in a finite (Galois) field over integers modulo a prime
  - nb. exponentiation takes $O((\log n)^3)$ operations (easy)
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
  - nb. factorization takes $O(e^{\log n \log \log n})$ operations (hard)

RSA Key Setup

- each user generates a public/private key pair by:
- selecting two large primes at random - p, q
- computing their system modulus $n=p.q$
  - note $\varnothing(n)=(p-1)(q-1)$
- selecting at random the encryption key e
  - where $1<e<\varnothing(n)$, $\gcd(e,\varnothing(n))=1$
- solve following equation to find decryption key d
  - $e.d \equiv 1 \mod \varnothing(n)$ and $0 \leq d \leq n$
- publish their public encryption key: PU={e,n}
- keep secret private decryption key: PR={d,n}

RSA Use

- to encrypt a message M the sender:
  - obtains **public key** of recipient PU={e,n}
  - computes: $C = M^e \bmod n$, where $0 \le M < n$
- to decrypt the ciphertext C the owner:
  - uses their private key PR={d,n}
  - computes: $M = C^d \bmod n$
- note that the message M must be smaller than the modulus n (block if needed)

Why RSA Works

- because of Euler's Theorem:
  - $a^{\varnothing(n)} \bmod n = 1$ where $\gcd(a,n)=1$
- in RSA have:
  - $n=p.q$
  - $\varnothing(n)=(p-1)(q-1)$
  - carefully chose e & d to be inverses mod $\varnothing(n)$
  - hence $e.d=1+k.\varnothing(n)$ for some k,

    $ed \equiv 1 \bmod \varnothing(n,$ so $d \equiv e^{-1} \bmod \varnothing(n)$
- hence :

$$C^{d} = M^{e.d} = M^{1+k.\varnothing(n)} = M^{1}.(M^{\varnothing(n)})^{k}$$

$$= M^{1}.(1)^{k} = M^{1} = M \bmod n$$

RSA Example - Key Setup

1. Select primes: p=17 & q=11
2. Compute $n = pq$ =17 x 11=187
3. Compute $ø(n)=(p–1)(q-1)$=16 x 10=160
4. Select e: gcd(e,160)=1; choose e=7
5. Determine d: de=1 mod 160 and d < 160 Value is d=23 since 23x7=161= 10x160+1
6. Publish public key PU={7,187}
7. Keep secret private key PR={23,187}

RSA Example - En/Decryption

- sample RSA encryption/decryption is:
- given message M = 88 (nb. 88<187)
- encryption:

  $C = 88^{7} \bmod 187 = 11$
- decryption:

  $M = 11^{23} \bmod 187 = 88$

Exponentiation

- can use the Square and Multiply Algorithm
- a fast, efficient algorithm for exponentiation
- concept is based on repeatedly squaring base
- and multiplying in the ones that are needed to compute the result
- look at binary representation of exponent
- only takes $O(\log_2 n)$ multiples for number n
  - eg. $7^5 = 7^4.7^1 = 3.7 = 10 \bmod 11$
  - eg. $3^{129} = 3^{128}.3^1 = 5.3 = 4 \bmod 11$

Exponentiation

```
c = 0; f = 1
for i = k down to 0
     do c = 2 x c
        f = (f x f) mod n
     if b_i == 1 then
        c = c + 1
        f = (f x a) mod n
 return f
```

Efficient Encryption

- encryption uses exponentiation to power e
- hence if e small, this will be faster
  - often choose e=65537 (2^{16}-1)
  - also see choices of e=3 or e=17
- but if e too small (eg e=3) can attack
  - using Chinese remainder theorem & 3 messages with different modulii
- if e fixed must ensure gcd(e,ø(n))=1
  - ie reject any p or q not relatively prime to e

Efficient Decryption

- decryption uses exponentiation to power d
  - this is likely large, insecure if not
- can use the Chinese Remainder Theorem (CRT) to compute mod p & q separately. then combine to get desired answer
  - approx 4 times faster than doing directly
- only owner of private key who knows values of p & q can use this technique

RSA Key Generation

- users of RSA must:
  - determine two primes at random - p, q
  - select either e or d and compute the other
- primes p, q must not be easily derived from modulus n=p.q
  - means must be sufficiently large
  - typically guess and use probabilistic test
- exponents e, d are inverses, so use Inverse algorithm to compute the other

RSA Security

- possible approaches to attacking RSA are:
  - brute force key search (infeasible given size of numbers)
  - mathematical attacks (based on difficulty of computing ø(n), by factoring modulus n)
  - timing attacks (on running of decryption)
  - chosen ciphertext attacks (given properties of RSA)

Factoring Problem

- mathematical approach takes 3 forms:
  - factor n=p.q, hence compute ø(n) and then d
  - determine ø(n) directly and compute d
  - find d directly
- currently believe all equivalent to factoring
  - have seen slow improvements over the years
    - as of May-05 best is 200 decimal digits (663) bit with LS
  - biggest improvement comes from improved algorithm
    - cf QS to GHFS to LS
  - currently assume 1024-2048 bit RSA is secure
    - ensure p, q of similar size and matching other constraints

Timing Attacks

- developed by Paul Kocher in mid-1990's
- exploit timing variations in operations
  - eg. multiplying by small vs large number
  - or IF's varying which instructions executed
- infer operand size based on time taken
- RSA exploits time taken in exponentiation
- countermeasures
  - use constant exponentiation time
  - add random delays
  - blind values used in calculations

Chosen Ciphertext Attacks

- RSA is vulnerable to a Chosen Ciphertext Attack (CCA)
- attackers chooses ciphertexts & gets decrypted plaintext back
- choose ciphertext to exploit properties of RSA to provide info to help cryptanalysis
- can counter with random pad of plaintext
- or use Optimal Asymmetric Encryption Padding (OASP)