Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
22
The Multistory Frames with Sidesway

Instructional Objectives

After reading this chapter the student will be able to

1. Identify the number of independent rotational degrees of freedom of a rigid frame.
2. Write appropriate number of equilibrium equations to solve rigid frame having more than one rotational degree of freedom.
3. Draw free-body diagram of multistory frames.
4. Analyse multistory frames with sidesway by the slope-deflection method.
5. Analyse multistory frames with sidesway by the moment-distribution method.

22.1 Introduction

In lessons 17 and 21, rigid frames having single independent member rotational ($\psi\left(=\frac{\Delta}{h}\right)$) degree of freedom (or joint translation Δ) is solved using slope-deflection and moment-distribution method respectively. However multistory frames usually have more than one independent rotational degree of freedom. Such frames can also be analysed by slope-deflection and moment-distribution methods. Usually number of independent member rotations can be evaluated by inspection. However if the structure is complex the following method may be adopted. Consider the structure shown in Fig. 22.1a. Temporarily replace all rigid joints of the frame by pinned joint and fixed supports by hinged supports as shown in Fig. 22.1b. Now inspect the stability of the modified structure. If one or more joints are free to translate without any resistance then the structure is geometrically unstable. Now introduce forces in appropriate directions to the structure so as to make it stable. The number of such externally applied forces represents the number of independent member rotations in the structure.

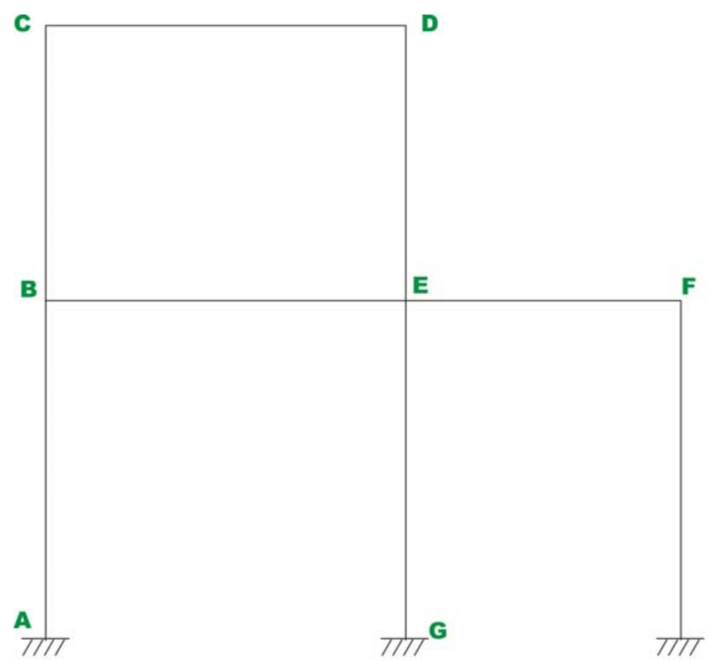

Fig. 22.1a Rigid frame

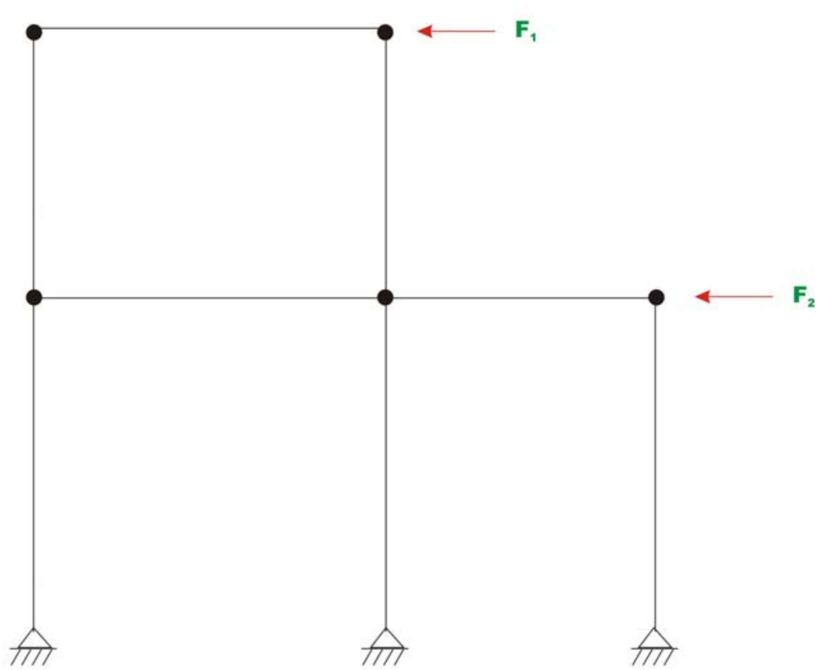

Fig. 22.1b Modified structure

In the modified structure Fig. 22.1b, two forces are required to be applied at level CD and level BF for stability of the structure. Hence there are two independent member rotations (ψ) that need to be considered apart from joint rotations in the analysis.

The number of independent rotations to be considered for the frame shown in Fig. 22.2a is three and is clear from the modified structure shown in Fig. 22.2b.

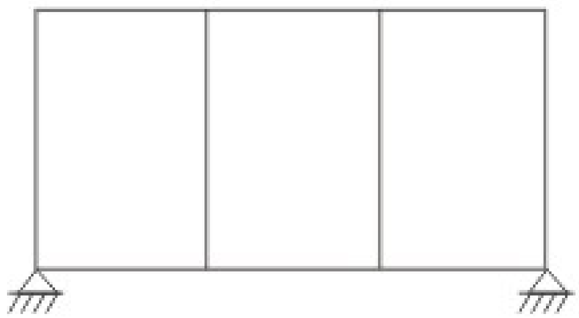

Figure 22.2a Rigid frame

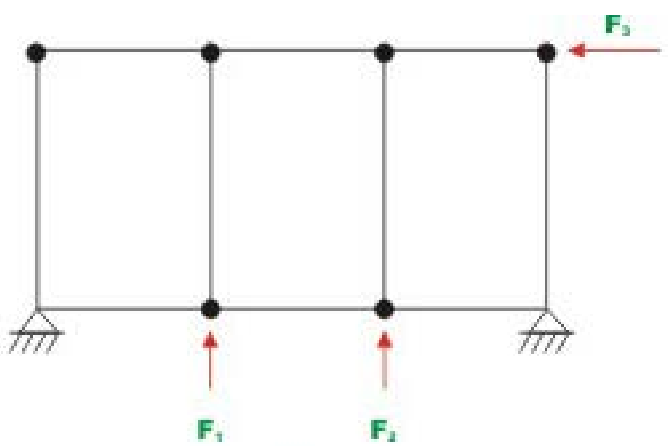

Figure 22.2b Modified structure

From the above procedure it is clear that the frame shown in Fig. 22.3a has three independent member rotations and frame shown in Fig 22.4a has two independent member rotations.

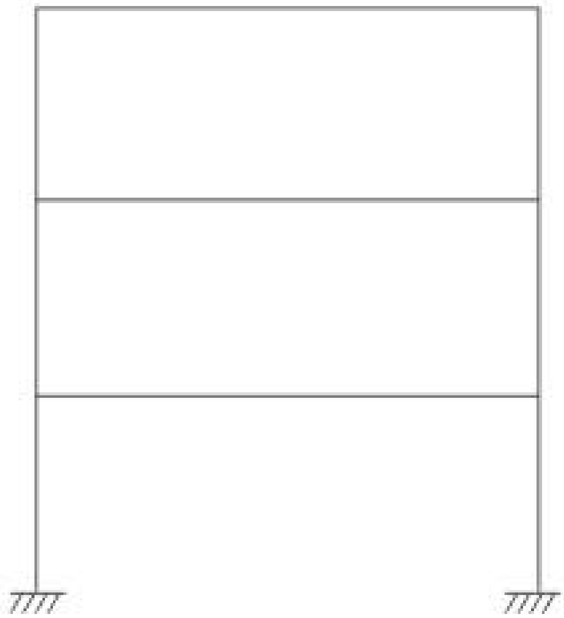

Figure 22.3a Rigid frame

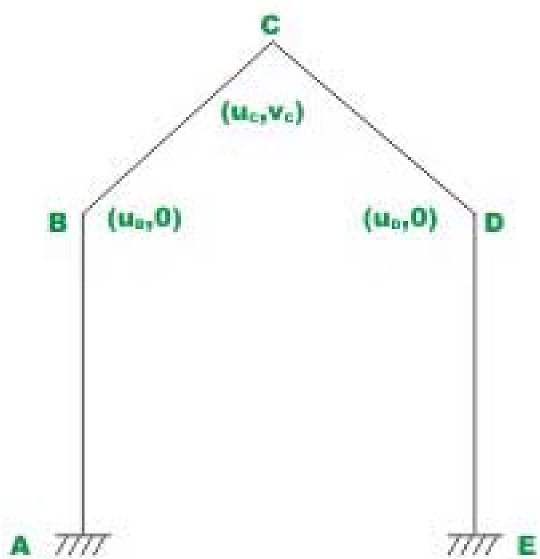

Figure 22.4a Gable frame

For the gable frame shown in Fig. 22.4a, the possible displacements at each joint are also shown. Horizontal displacement is denoted by u and vertical displacement is denoted by v. Recall that in the analysis, we are not considering the axial deformation. Hence at B and D only horizontal deformation is possible and joint C can have both horizontal and vertical deformation. The displacements u_B, u_C, u_D and u_D should be such that the lengths BC and CD must not change as the axial deformation is not considered. Hence we can have only two independent translations. In the next section slope-deflection method as applied to multistoried frame is discussed.

22.2 Slope-deflection method

For the two story frame shown in Fig. 22.5, there are four joint rotations $(\theta_B, \theta_C, \theta_D \; and \; \theta_E)$ and two independent joint translations (sidesway) Δ_1 at the level of CD and Δ_2 at the level of BE.

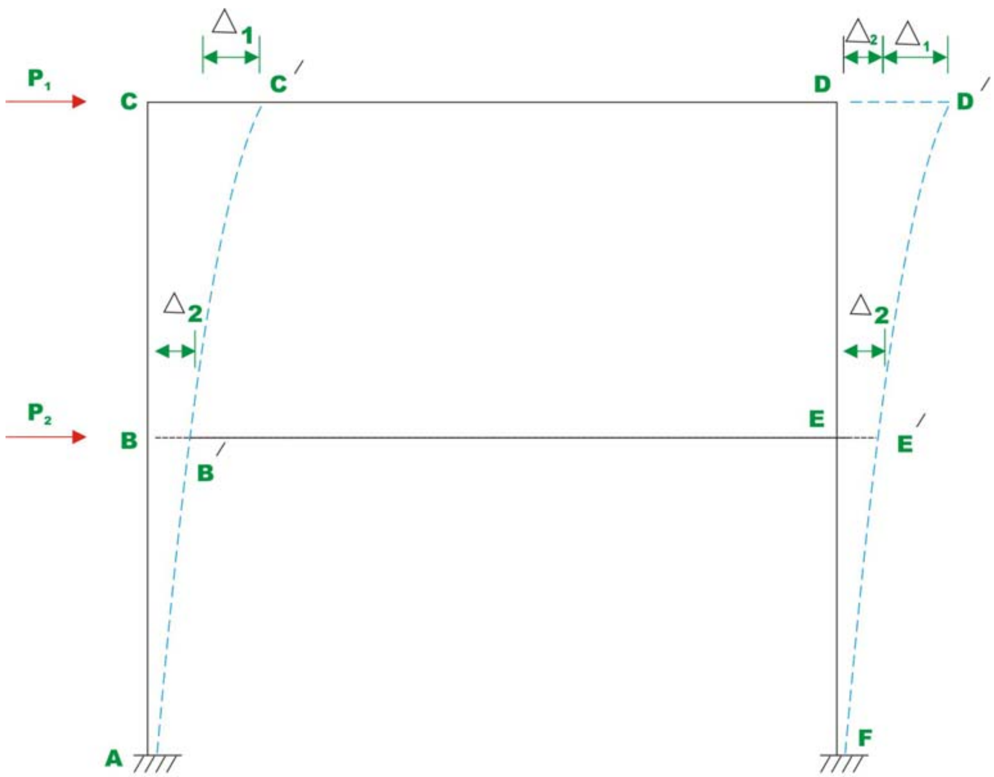

Fig.22.5 Two story frame.

Six simultaneous equations are required to evaluate the six unknowns (four rotations and two translations). For each of the member one could write two slope-deflection equations relating beam end moments to (i) externally applied loads and (ii) displacements (rotations and translations). Four of the required six equations are obtained by considering the moment equilibrium of joint B, C, D and E respectively. For example,

$$\sum M_B = 0 \qquad \Rightarrow M_{BA} + M_{BC} + M_{BE} = 0 \tag{22.1}$$

The other two equations are obtained by considering the force equilibrium of the members. Thus, the shear at the base of all columns for any story must be equal to applied load. Thus $\sum F_X = 0$ at the base of top story gives (ref. Fig. 22.6)

$$P_1 - H_C - H_D = 0 \tag{22.2}$$

Similarly $\sum F_X = 0$ at the base of frame results in

$$P_1 + P_2 - H_A - H_F = 0 \tag{22.3}$$

Thus we get six equations in six unknowns. Solving the above six equations all the unknowns are evaluated. The above procedure is explained in example 22.1.

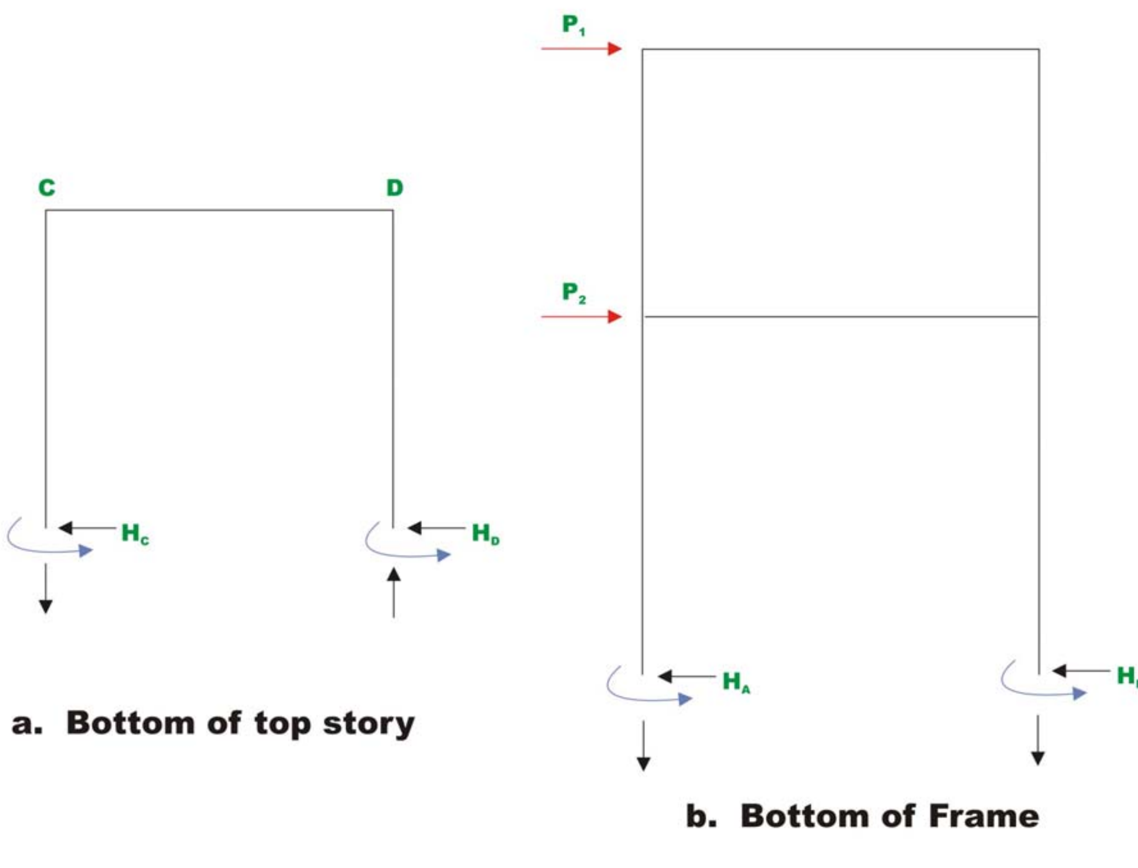

Fig. 22.6

Example 22.1

Analyse the two story rigid frame shown in Fig. 22.7a by the slope-deflection method. Assume EI to be constant for all members.

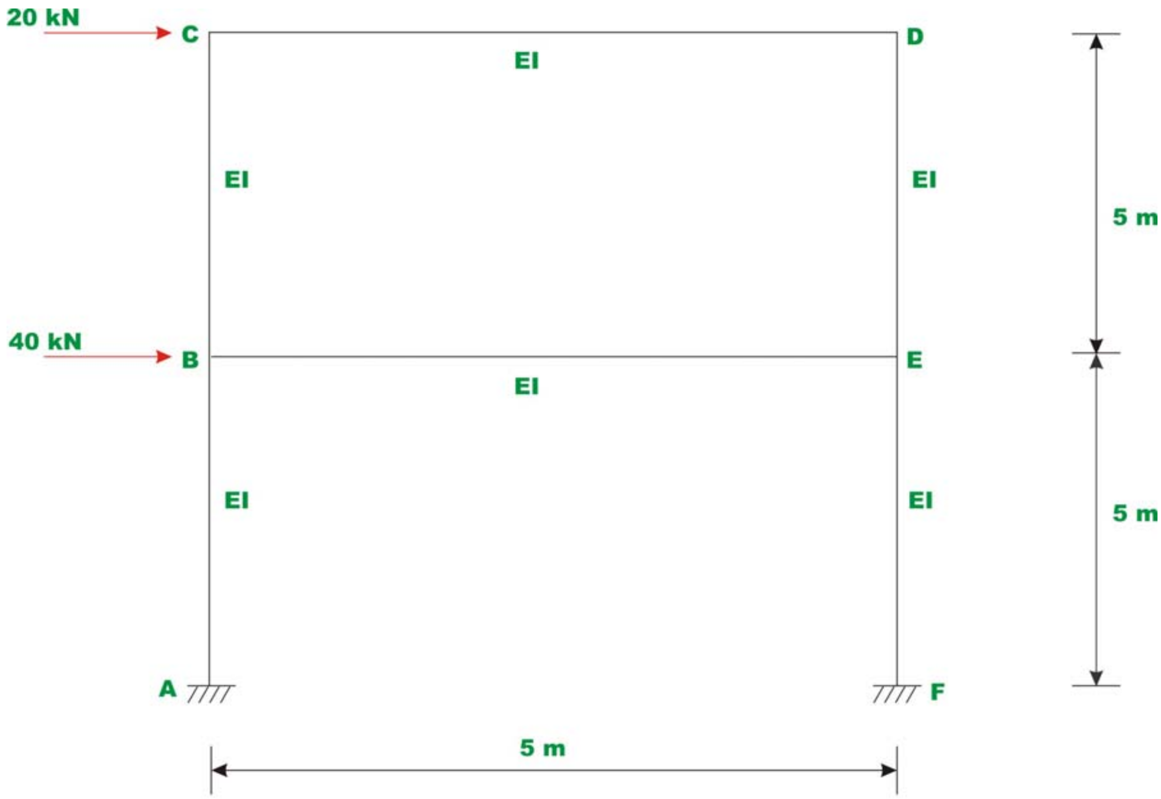

Fig.22.7a Example 22.1

In this case all the fixed end moments are zero. The members AB and EF undergo rotations $\psi_2 = -\frac{\Delta_2}{5}$ (negative as it is clockwise) and member BC and ED undergo rotations $\psi_1 = -\frac{\Delta_1}{5}$. Now writing slope-deflection equations for 12 beam end moments.

$$M_{AB} = 0 + \frac{2EI}{5}\left[2\theta_A + \theta_B - 3\psi_2\right] \qquad \theta_A = 0; \qquad \psi_2 = -\frac{\Delta_2}{5}$$

$$M_{AB} = 0.4EI\theta_B + 0.24EI\Delta_2$$

$$M_{BA} = 0.8EI\theta_B + 0.24EI\Delta_2$$

$$M_{BC} = 0.8EI\theta_B + 0.4EI\theta_C + 0.24EI\Delta_1$$

$$M_{CB} = 0.8EI\theta_C + 0.4EI\theta_B + 0.24EI\Delta_1$$

$$M_{BE} = 0.8EI\theta_B + 0.4EI\theta_E$$

$$M_{EB} = 0.8EI\theta_E + 0.4EI\theta_B$$

$$M_{CD} = 0.8EI\theta_C + 0.4EI\theta_D$$

$$M_{DC} = 0.8EI\theta_D + 0.4EI\theta_C$$

$$M_{DE} = 0.8EI\theta_D + 0.4EI\theta_E + 0.24EI\Delta_1$$

$$M_{ED} = 0.8EI\theta_E + 0.4EI\theta_D + 0.24EI\Delta_1$$

$$M_{EF} = 0.8EI\theta_E + 0.24EI\Delta_2$$

$$M_{FE} = 0.4EI\theta_E + 0.24EI\Delta_2 \qquad (1)$$

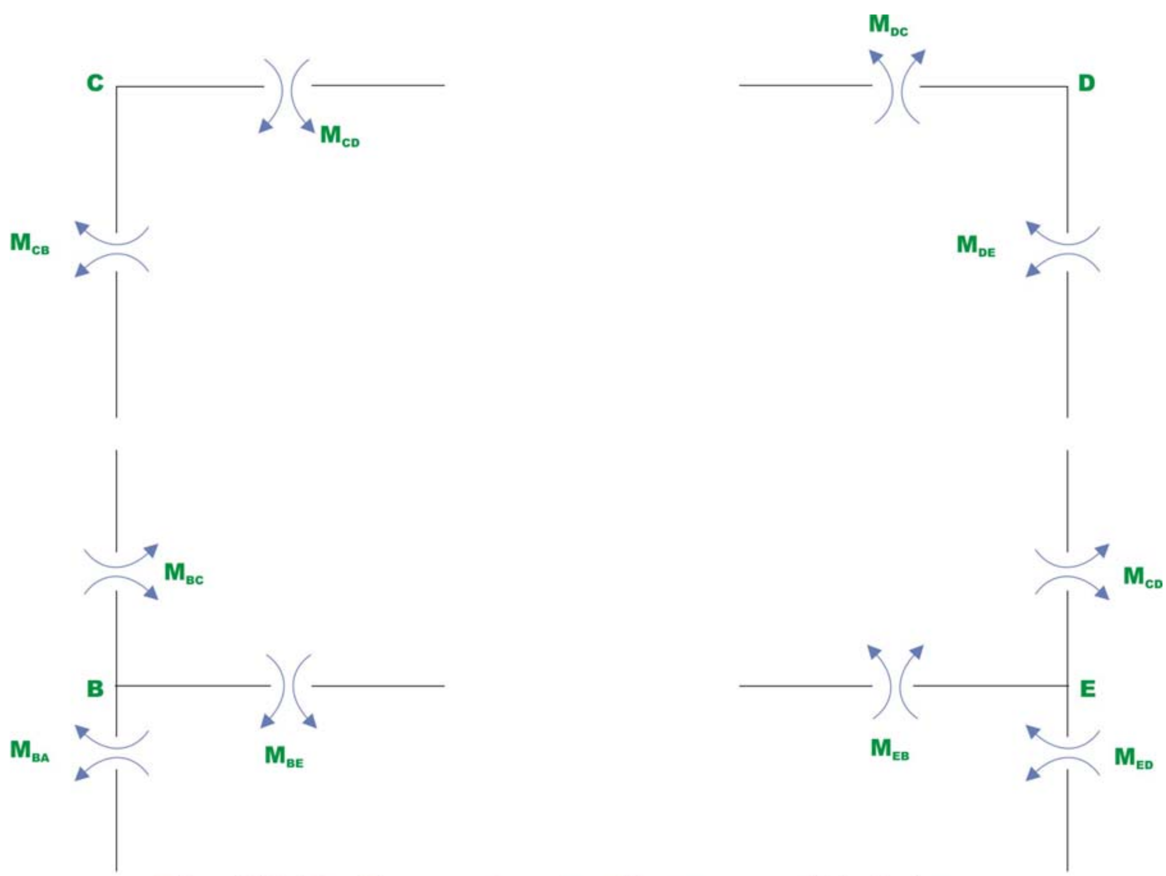

Fig. 22.7c Free - body diagram of joints

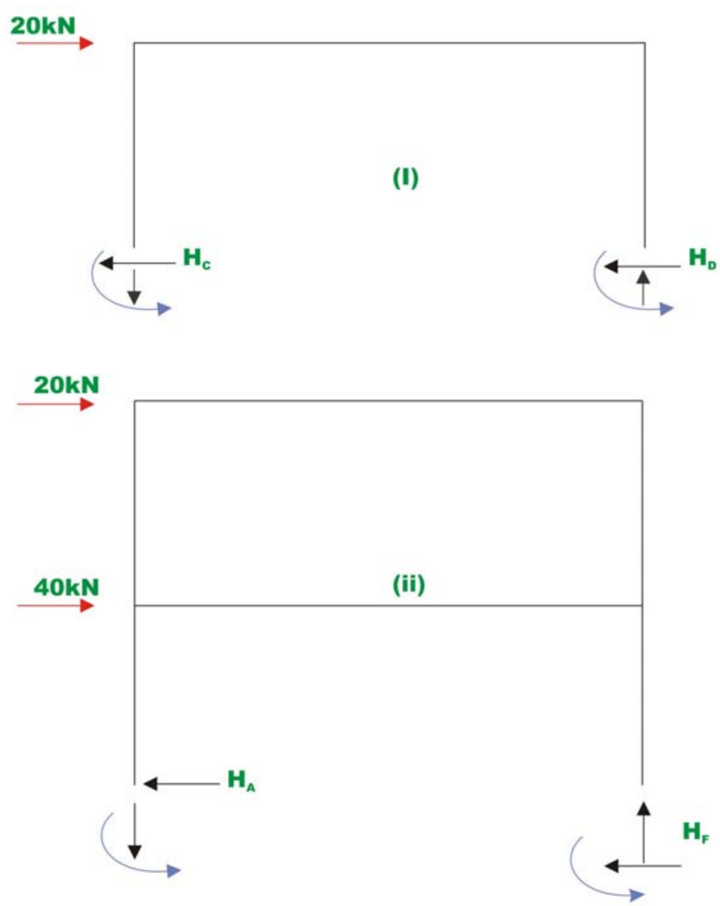

Fig.22.7d Free - body diagram

Moment equilibrium of joint B, C, D and E requires that (vide Fig. 22.7c).

$$
\begin{aligned}
&M_{BA} + M_{BC} + M_{BE} = 0 \\
&M_{CB} + M_{CD} = 0 \\
&M_{DC} + M_{DE} = 0 \\
&M_{EB} + M_{ED} + M_{EF} = 0
\end{aligned} \qquad (2)
$$

The required two more equations are written considering the horizontal equilibrium at each story level. *i.e.* $\sum F_X = 0$ (vide., Fig. 22.7d). Thus,

$$H_C + H_D = 20$$

$$H_A + H_F = 60 \qquad (3)$$

Considering the equilibrium of column AB, EF, BC and ED, we get (vide 22.7c)

$$H_C = \frac{M_{BC} + M_{CB}}{5}$$

$$H_D = \frac{M_{DE} + M_{ED}}{5}$$

$$H_A = \frac{M_{AB} + M_{BA}}{5}$$

$$H_F = \frac{M_{EF} + M_{FE}}{5} \qquad (4)$$

Using equation (4), equation (3) may be written as,

$$M_{BC} + M_{CB} + M_{DE} + M_{ED} = 100$$

$$M_{AB} + M_{BA} + M_{EF} + M_{FE} = 300 \qquad (5)$$

Substituting the beam end moments from equation (1) in (2) and (5) the required equations are obtained. Thus,

$$2.4\theta_B + 0.4\theta_C + 0.4\theta_E + 0.24\Delta_1 + 0.24\Delta_2 = 0$$

$$1.6\theta_C + 0.4\theta_D + 0.4\theta_B + 0.24\Delta_1 = 0$$

$$1.6\theta_D + 0.4\theta_C + 0.4\theta_E + 0.24\Delta_1 = 0$$

$$2.4\theta_E + 0.4\theta_B + 0.4\theta_D + 0.24\Delta_1 + 0.24\Delta_2 = 0$$

$$1.2\theta_B + 1.2\theta_C + 1.2\theta_D + 1.2\theta_E + 0.96\Delta_1 = 100$$

$$1.2\theta_B + 1.2\theta_E + +0.96\Delta_2 = 300 \qquad (6)$$

Solving above equations, yields

$$\theta_B = \frac{-65.909}{EI}; \quad \theta_C = \frac{-27.273}{EI}; \quad \theta_E = \frac{-27.273}{EI}; \quad \theta_D = \frac{-65.909}{EI};$$
$$\Delta_1 = \frac{337.12}{EI}; \quad \Delta_2 = \frac{477.27}{EI} \qquad (7)$$

Substituting the above values of rotations and translations in equation (1) beam end moments are evaluated. They are,

$$M_{AB} = 88.18 \text{ kN.m} \; ; \; M_{BA} = 61.81 \text{ kN.m}$$

$$M_{BC} = 17.27 \text{ kN.m} \; ; \; M_{CB} = 32.72. \text{ kN.m}$$

$$M_{BE} = -79.09 \text{ kN.m} \; ; \; M_{EB} = -79.09 \text{ kN.m}$$

$$M_{CD} = -32.72 \text{ kN.m} \; ; \; M_{DC} = -32.72 \text{ kN.m}$$

$$M_{DE} = 32.72 \text{ kN.m} \; ; \; M_{ED} = 17.27 \text{ kN.m}$$

$$M_{EF} = 61.81 \text{ kN.m} \; ; \; M_{FE} = 88.18 \text{ kN.m}$$

22.3 Moment-distribution method

The two-story frame shown in Fig. 22.8a has two independent sidesways or member rotations. Invoking the method of superposition, the structure shown in Fig. 22.8a is expressed as the sum of three systems;

1) The system shown in Fig. 22.8b, where in the sidesway is completely prevented by introducing two supports at E and D. All external loads are applied on this frame.
2) System shown in Fig. 22.8c, wherein the support E is locked against sidesway and joint C and D are allowed to displace horizontally. Apply arbitrary sidesway Δ'_1 and calculate fixed end moments in column BC and DE. Using moment-distribution method, calculate beam end moments.
3) Structure shown in Fig. 22.8d, the support D is locked against sidesway and joints B and E are allowed to displace horizontally by removing the support at E. Calculate fixed end moments in column AB and EF for an arbitrary sidesway Δ'_2 as shown the in figure. Since joint displacement as known beforehand, one could use the moment-distribution method to analyse the frame.

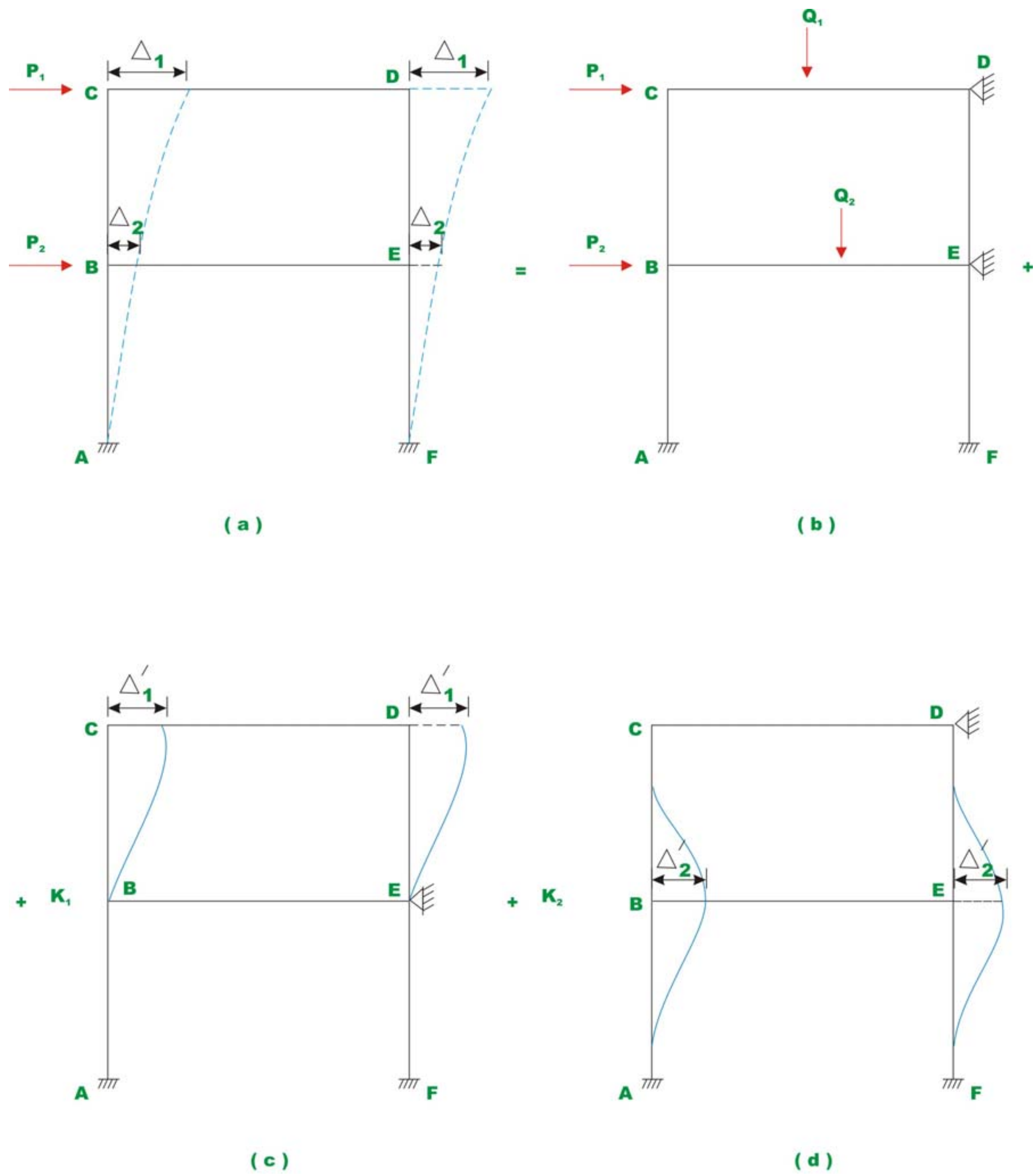

Fig.22.8 Two - story frame

All three systems are analysed separately and superposed to obtain the final answer. Since structures 22.8c and 22.8d are analysed for arbitrary sidesway Δ'_1 and Δ'_2 respectively, the end moments and the displacements of these two analyses are to be multiplied by constants k_1 and k_2 before superposing with the results obtained in Fig. 22.8b. The constants k_1 and k_2 must be such that

$$k_1\,\Delta'_1 = \Delta_1 \quad \text{and} \quad k_2\,\Delta'_2 = \Delta_2 \,. \tag{22.4}$$

The constants k_1 and k_2 are evaluated by solving shear equations. From Fig. 22.9, it is clear that the horizontal forces developed at the beam level CD in Fig. 22.9c and 22.9d must be equal and opposite to the restraining force applied at the restraining support at D in Fig. 22.9b. Thus,

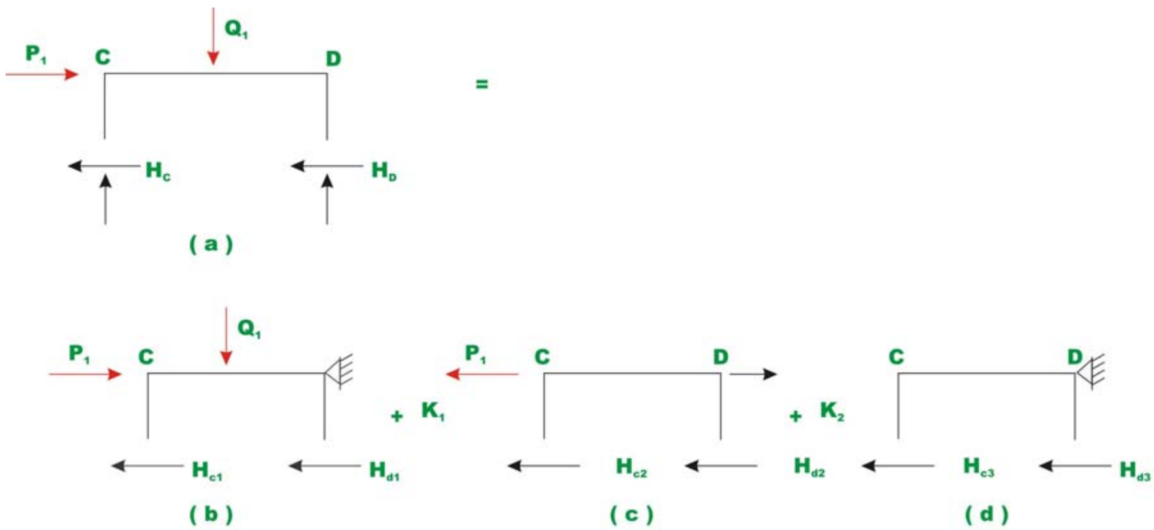

Fig.22.9 The free body diagram at top story.

$$k_1\left(H_{C2} + H_{D2}\right) + k_2\left(H_{C3} + H_{D3}\right) = P_1 \tag{23.5}$$

From similar reasoning, from Fig. 22.10, one could write,

$$k_1\left(H_{A2} + H_{F2}\right) + k_2\left(H_{A3} + H_{F3}\right) = P_2 \tag{23.6}$$

Solving the above two equations, k_1 and k_2 are calculated.

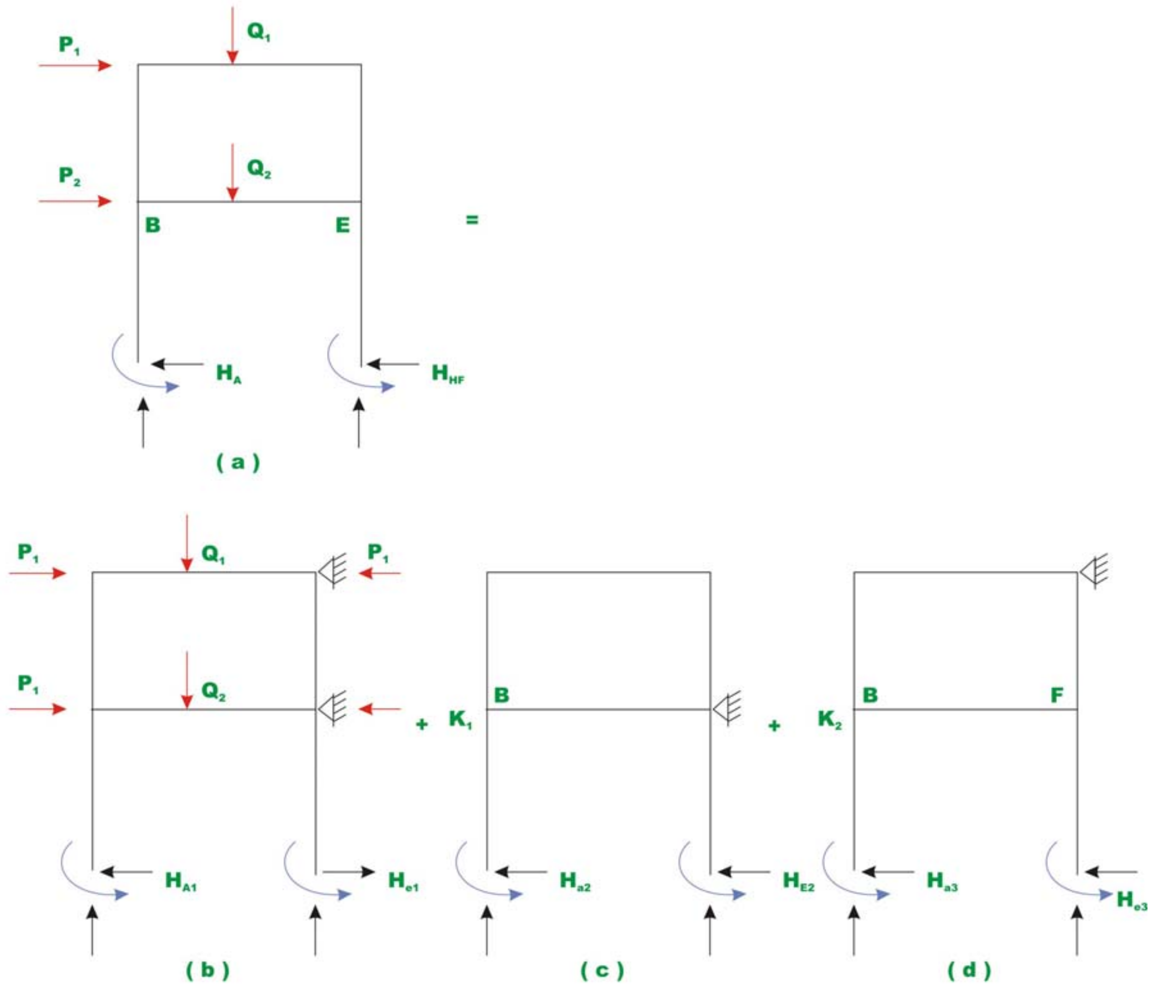

Fig.22.10 Free body diagram at the base of Frame.

Example 22.2

Analyse the rigid frame of example 22.1 by the moment-distribution method.

Solution:

First calculate stiffness and distribution factors for all the six members.

$$\begin{aligned} &K_{BA} = 0.20EI; \quad &K_{BC} = 0.20EI; \quad &K_{BE} = 0.20EI; \\ &K_{CB} = 0.20EI; \quad &K_{CD} = 0.20EI; \quad & \\ &K_{DC} = 0.20EI; \quad &K_{DE} = 0.20EI; \quad & \\ &K_{EB} = 0.20EI; \quad &K_{ED} = 0.20EI; \quad &K_{EF} = 0.20EI \end{aligned} \tag{1}$$

Joint B: $\sum K = 0.60EI$

$$DF_{BA} = 0.333; \qquad DF_{BC} = 0.333; \qquad DF_{BE} = 0.333$$

Joint C : $\sum K = 0.40EI$

$DF_{CB} = 0.50$; $DF_{CD} = 0.50$

Joint D : $\sum K = 0.40EI$

$DF_{DC} = 0.50$; $DF_{DE} = 0.50$

Joint E : $\sum K = 0.60EI$

$DF_{EB} = 0.333$; $DF_{ED} = 0.333$; $DF_{EF} = 0.333$ (2)

The frame has two independent sidesways: Δ_1 to the right of CD and Δ_2 to the right of BE. The given problem may be broken in to three systems as shown in Fig.22.11a.

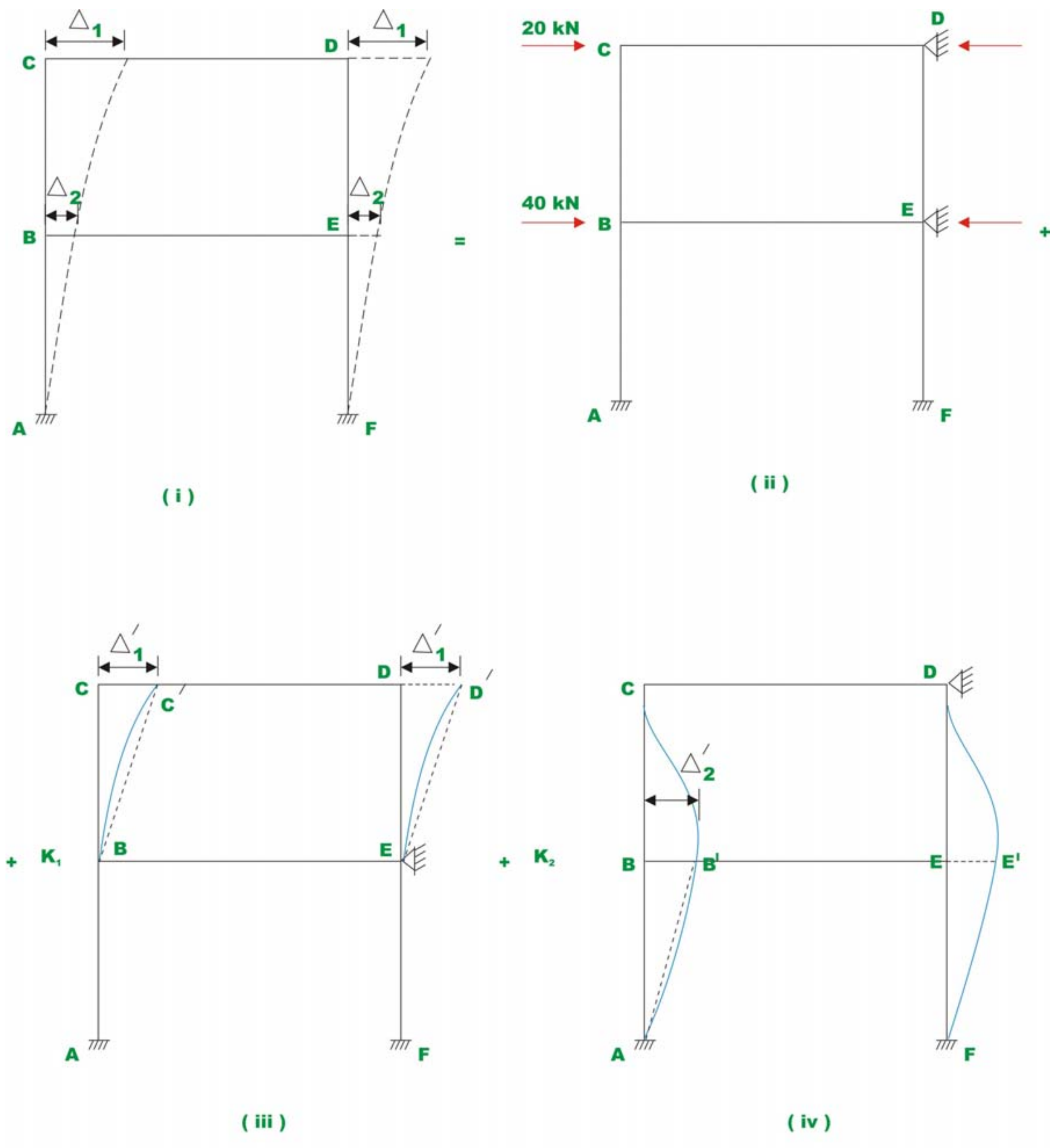

Fig. 22.11a Example 22.2

In the first case, when the sidesway is prevented [Fig. 22.10a (ii)], the only internal forces induced in the structure being $20\,kN$ and $40\,kN$ axial forces in member CD and BE respectively. No bending moment is induced in the structure. Thus we need to analyse only (iii) and (iv).

Case I :

Moment-distribution for sidesway Δ'_1 at beam CD [Fig. 22.1qa (iii)]. Let the arbitrary sidesway be $\Delta'_1 = \frac{25}{EI}$. Thus the fixed end moment in column CB and DE due to this arbitrary sidesway is

$$M_{BC}^{F} = M_{CB}^{F} = \frac{6EI\Delta'_1}{L^2} = \frac{6EI}{25} \times \frac{25}{EI} = +6.0 \quad \text{kN.m}$$

$$M_{ED}^{F} = M_{DE}^{F} = +6.0 \quad \text{kN.m} \qquad (3)$$

Now moment-distribution is carried out to obtain the balanced end moments. The whole procedure is shown in Fig. 22.11b. Successively joint D, C, B and E are released and balanced.

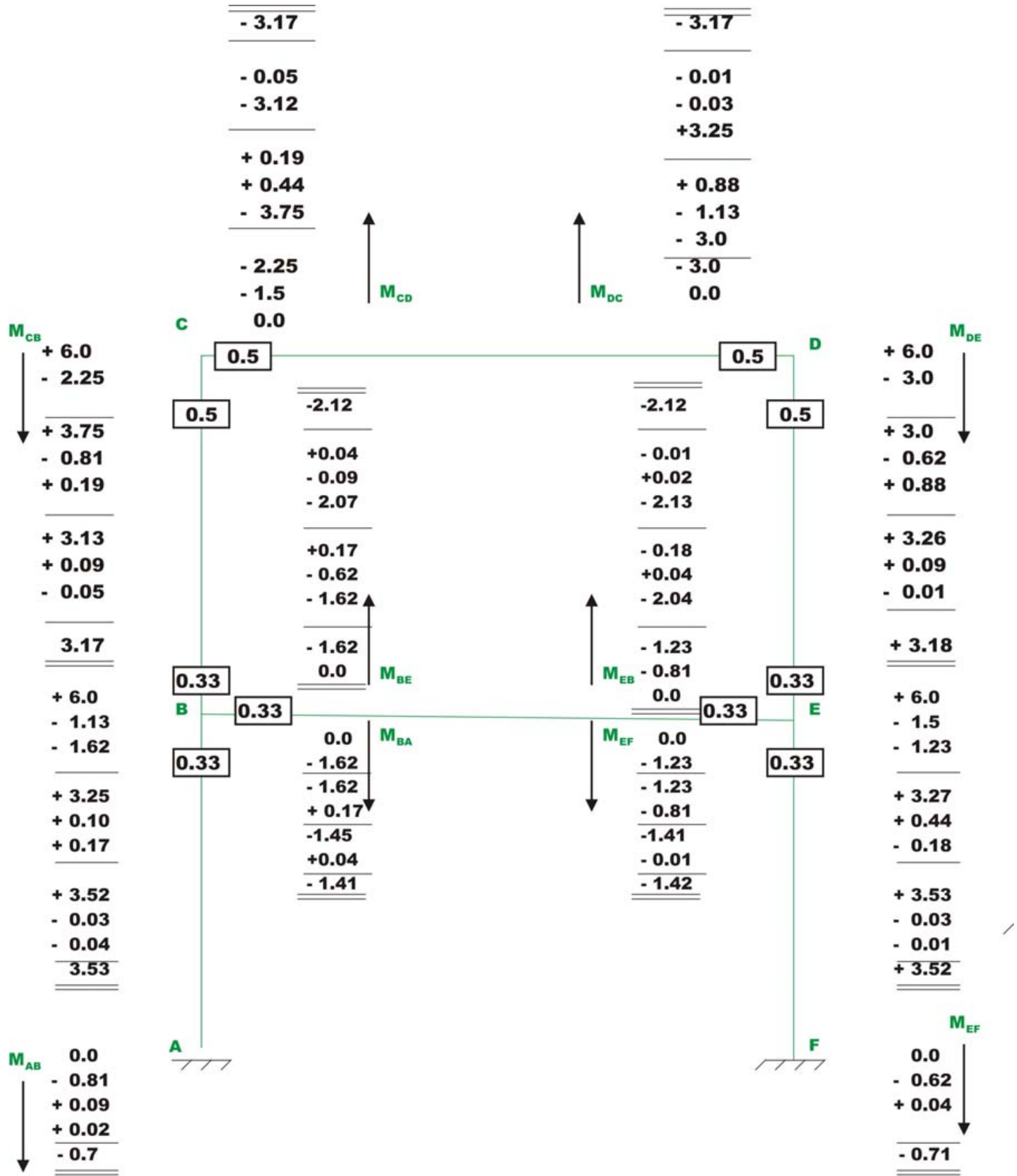

Fig. 22.11b Moment distribution for known sidesway at top story

From the free body diagram of the column shown in Fig. 22.11c, the horizontal forces are calculated. Thus,

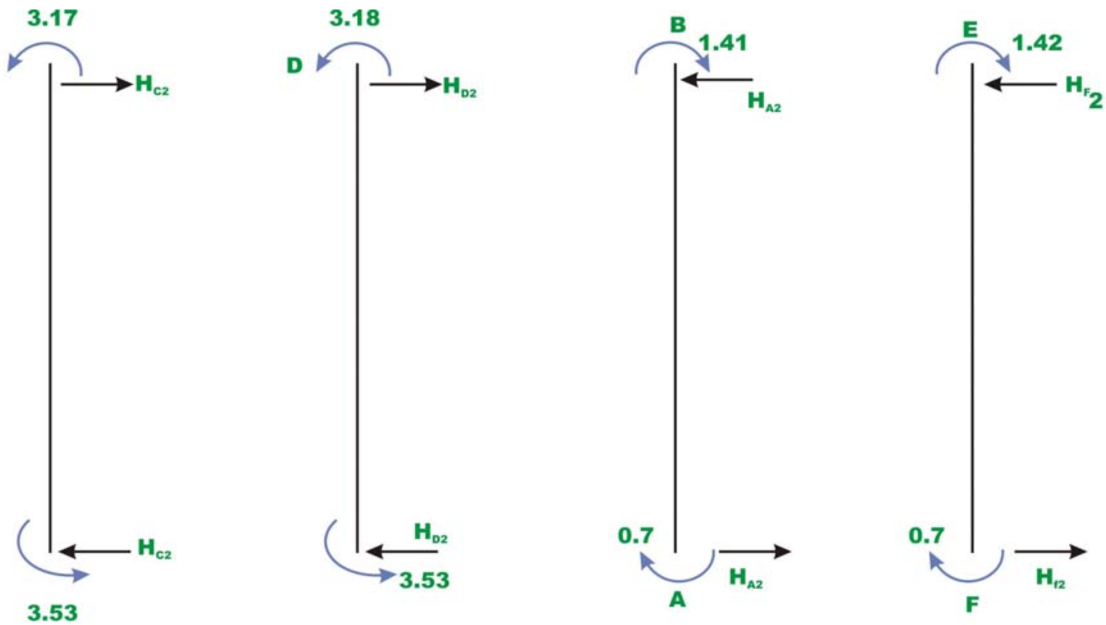

Fig.22.11c Free - body diagrams of Columns for applied load

$$H_{C2} = \frac{3.53 + 3.17}{5} = 1.34 \quad \text{kN}; \qquad H_{D2} = 1.34 \quad \text{kN}$$

$$H_{A2} = \frac{-0.70 - 1.41}{5} = -0.42 \quad \text{kN}; \qquad H_{F2} = -0.42 \quad \text{kN} \qquad (4)$$

Case II **:**

Moment-distribution for sidesway Δ'_2 at beam BE [Fig. 22.11a (iv)]. Let the arbitrary sidesway be $\Delta'_2 = \frac{25}{EI}$

Thus the fixed end moment in column AB and EF due to this arbitrary sidesway is

$$M_{AB}^F = M_{BA}^F = \frac{6EI\Delta'_2}{L^2} = \frac{6EI}{25} \times \frac{25}{EI} = +6.0 \quad \text{kN.m}$$

$$M_{FE}^F = M_{EF}^F = +6.0 \quad \text{kN.m} \qquad (5)$$

Moment-distribution is carried out to obtain the balanced end moments as shown in Fig. 22.11d. The whole procedure is shown in Fig. 22.10b. Successively joint D, C, B and E are released and balanced.

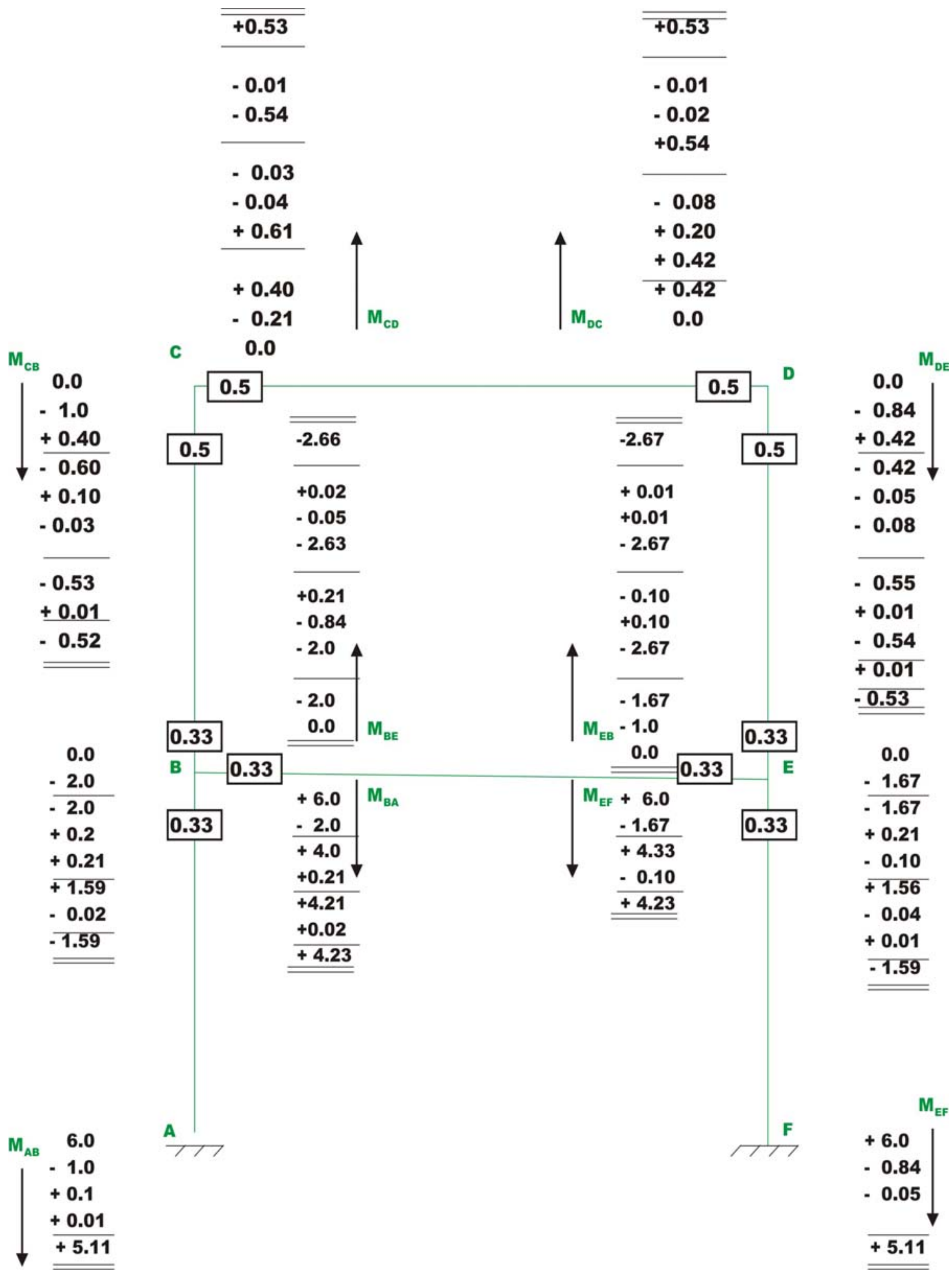

Fig. 22.11d Moment distribution for known sidesway at bottom story

From the free body diagram of the column shown in Fig. 22.11e, the horizontal forces are calculated. Thus,

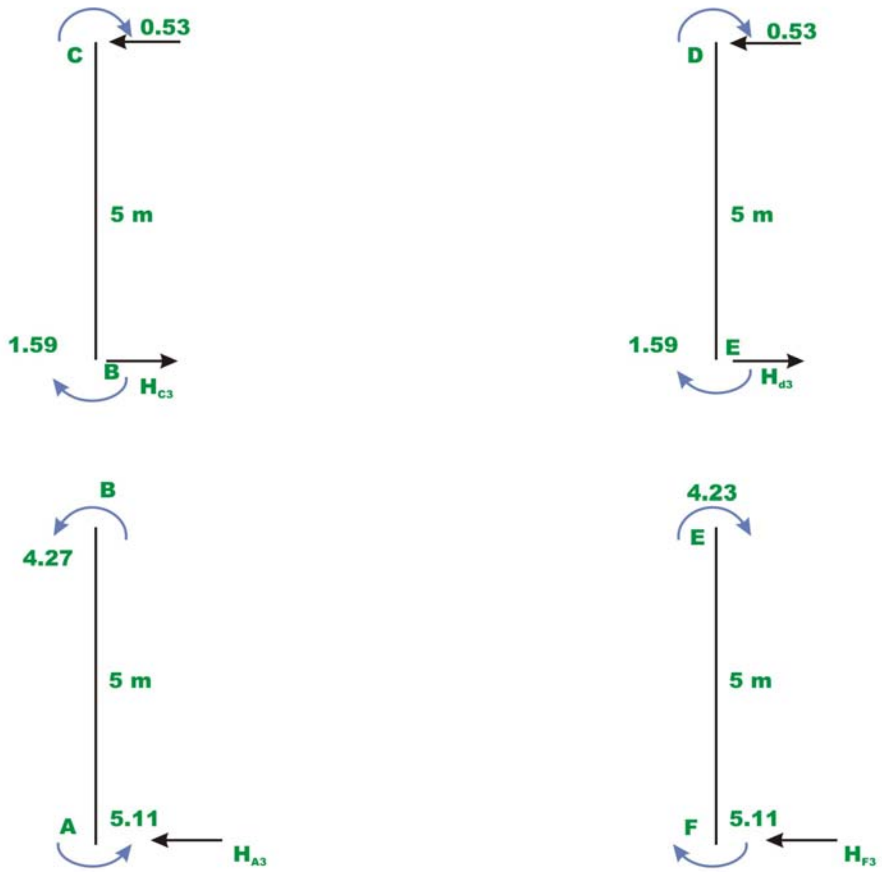

Figure 22.11e Free - body diagrams of Columns for arbitrary known sidesway

$$H_{C3} = \frac{-1.59 - 0.53}{5} = -0.42 \text{ kN}; \qquad H_{D3} = -0.42 \text{ kN}$$
$$H_{A3} = \frac{5.11 + 4.23}{5} = 1.86 \text{ kN}; \qquad H_{F3} = 1.86 \text{ kN} \qquad (6)$$

For evaluating constants k_1 and k_2, we could write, (see Fig. 22.11a, 22.11c and 22.11d).

$$k_1(H_{C2} + H_{D2}) + k_2(H_{C3} + H_{D3}) = 20$$
$$k_1(H_{A2} + H_{F2}) + k_2(H_{A3} + H_{F3}) = 60$$
$$k_1(1.34 + 1.34) + k_2(-0.42 - 0.42) = 20$$
$$k_1(-0.42 - 0.42) + k_2(1.86 + 1.86) = 60$$

$$k_1(1.34) + k_2(-0.42) = 10$$
$$k_1(-0.42) + k_2(1.86) = 30$$

Solving which, $k_1 = 13.47$ $\qquad$ $k_2 = 19.17$ (7)

Thus the final moments are,

$$M_{AB} = 88.52 \text{ kN.m} \; ; \; M_{BA} = 62.09 \text{ kN.m}$$

$$M_{BC} = 17.06 \text{ kN.m} \; ; \; M_{CB} = 32.54. \text{ kN.m}$$

$$M_{BE} = -79.54 \text{ kN.m} \; ; M_{EB} = -79.54 \text{ kN.m}$$

$$M_{CD} = -32.54 \text{ kN.m} \; ; M_{DC} = -32.54 \text{ kN.m}$$

$$M_{DE} = 32.54 \text{ kN.m} \; ; \; M_{ED} = 17.06 \text{ kN.m}$$

$$M_{EF} = 62.09 \text{ kN.m} \; ; \; M_{FE} = 88.52 \text{ kN.m} \qquad (8)$$

Summary

A procedure to identify the number of independent rotational degrees of freedom of a rigid frame is given. The slope-deflection method and the moment-distribution method are extended in this lesson to solve rigid multistory frames having more than one independent rotational degrees of freedom. A multistory frames having side sway is analysed by the slope-deflection method and the moment-distribution method. Appropriate number of equilibrium equations is written to evaluate all unknowns. Numerical examples are explained with the help of free-body diagrams.