Artificial Intelligence: Foundations & Applications

Introduction to Constraint Satisfaction Problem

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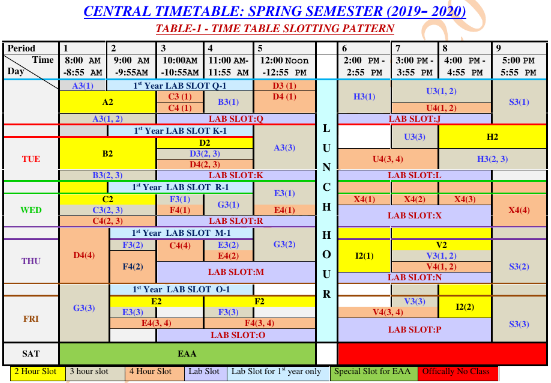
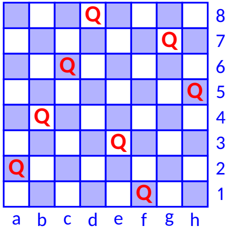
Examples of CSP

- Crossword puzzle
- N-queens on chess board
- Knapsack
- Assembly scheduling
- Operations research
- Map coloring
- Time tabling
- Airline/train scheduling
- Cryptic puzzle
- Boolean satisfiability
- Car sequencing
- Scene labeling
- etc.





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  - **A set of *decision* variables** $x_1, x_2, \ldots, x_n$

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  - **A set of optimization functions (typically** $\min, \max$**)** $O_1, O_2, \ldots, O_p$

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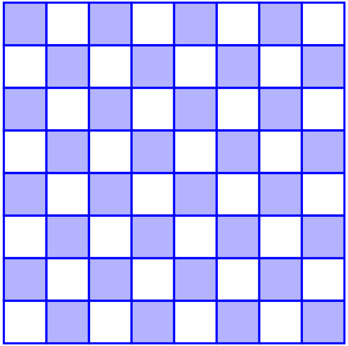
- Cost function for optimization (optional)
  - A set of optimization functions (typically $\min, \max$) $O_1, O_2, \ldots, O_p$
- Solution
  - A consistent assignment of domain values to each variable so that all constraints are satisfied and the optimization criteria (if any) are met.

N-Queens

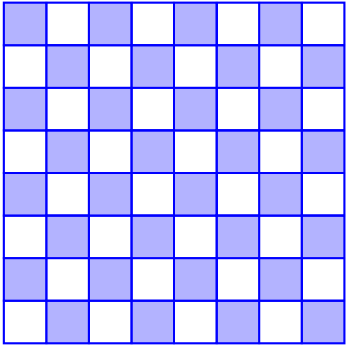


Need to place N-queens on this board

Rules:

- No queens are attacking each other

N-Queens



Need to place N-queens on this board

Rules:

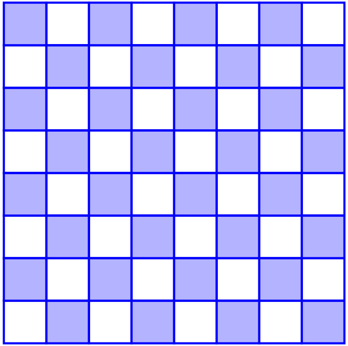
- No queens are attacking each other

- Variables: x_{ij} - queen is in cell (i,j),
- Domains: $D_{ij} \in \{0,1\}$
- Constraints: $\sum_i x_{ij} = 1, \sum_j x_{ij} = 1, \sum_{i,j} x_{ij} = N,$

  $x_{ij} + x_{(i+k)(j+k)} \leq 1, \quad x_{ij} + x_{(i+k)(j-k)} \leq 1,$

  k is in appropriate range
- Search space $2^{64} = 18,446,744,073,709,551,616$

N-Queens (alternative model)



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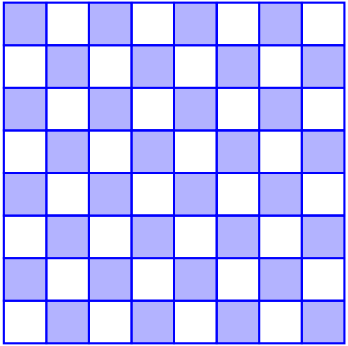

Need to place N-queens on this board

Rules:

- No queens are attacking each other

- Variables: x_i
- Domains: $D_i \in \{1, 2, \ldots, 8\}$
- Constraints: $\ldots$
- Search space $8^8 = 16,777,216$

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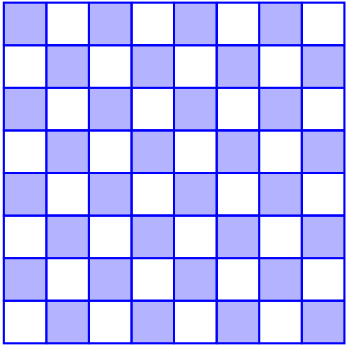

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- No queens are attacking each other

- Variables: x_i
- Domains: $D_i \in \{1, 2, \ldots, 8\}$
- Constraints: $\ldots$
- Search space $8^8 = 16,777,216$

Other variants:
- At least a queen on the main diagonal
- Two queens on the two main diagonals
- Enumeration of all solutions

Examination schedule

Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6

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Is it possible to conduct all these exams in 3 days assuming one exam per day?

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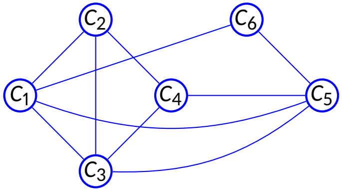

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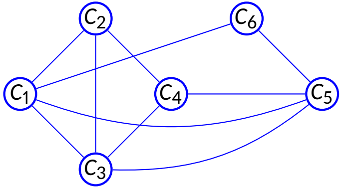

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- Variables: x_i - slot for subject C_i
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- Constraints: $x_1 \neq x_2, x_1 \neq x_3, \ldots$

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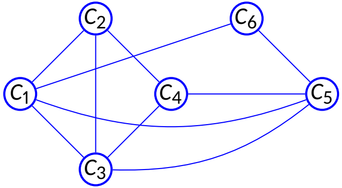

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Graph coloring problem.

Airport gate scheduling

Flight	Arrv. time	Dept. time
F1	0715	0815
F2	0800	0900
F3	0830	0930
F4	0845	0945
F5	0915	1015
F6	0845	0945

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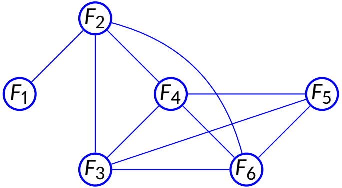

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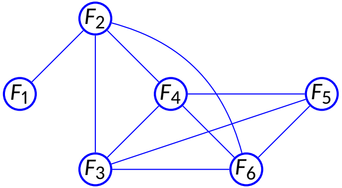

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- Variables: x_i - slot for Flight F_i
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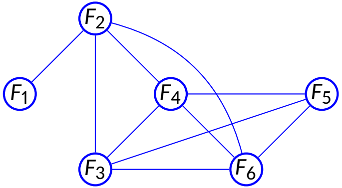

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Interval Graphs.

Cryptarithmetic

	S	E	N	D
+	M	O	R	E
M	O	N	E	Y

Cryptarithmetic

$$\begin{array}{cccccc} & S & E & N & D \\ + & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$

- Variables: $S, E, N, D, M, O, R, Y,$
- Domains: $D_i \in \{0, 1, \ldots, 9\}$
- Constraints: All different, $10 \times M + O = S + M + C_{1000}, \ldots$

Cryptarithmetic

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MiniZinc implementation:

```
include "alldifferent.mzn";

var 1..9: S; var 0..9: E; var 0..9: N; var 0..9: D;
var 1..9: M; var 0..9: O; var 0..9: R; var 0..9: Y;

constraint
              1000 * S + 100 * E + 10 * N + D
            + 1000 * M + 100 * O + 10 * R + E
= 10000 * M + 1000 * O + 100 * N + 10 * E + Y;

constraint alldifferent([S,E,N,D,M,O,R,Y]);

solve satisfy;
```

Knapsack

- **There are n items namely, $O_1, O_2, \ldots, O_n$. Item O_i weighs w_i and provides profit of p_i. Target is to select a subset of the items such that the total weight of the items does not exceed W and profit is maximized.**

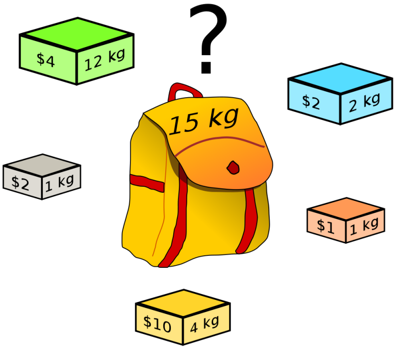

image source: Wikipedia

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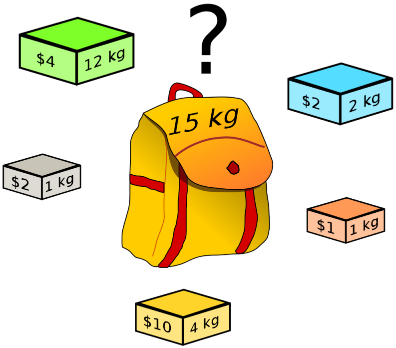
  - Variables: x_i - selection of ith item
  - Domains: $\{0, 1\}$
  - Constraints: $\sum_i x_i \times w_i \leq W$
  - Optimization function: $\sum_i x_i \times p_i$



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Warehouse planning

- **There are n possible locations to setup warehouses (W) which will deliver goods to m customers (C). Cost to setup W_j warehouse is f_j. Customer C_i has a demand of d_i which needs to fulfilled by the warehouses. Delivery cost per unit item from W_j to C_i is c_{ji}. Target is to minimize total cost to serve the required demands.**

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  - **Variables:** x_j **- warehouse location,** y_{ji} **- amount served by** W_j **to** C_i
  - **Domains:** $x_j \in \{0, 1\}$, $y_{ji} \in \{0, \infty\}$
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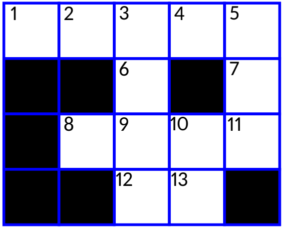
  - **Optimization function:** $\sum_i x_j \times f_j + \sum_{i,j} c_{ji} \times y_{ji}$

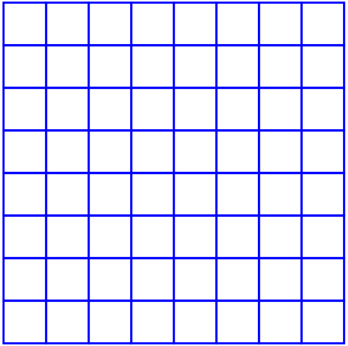
Crossword puzzle



Fill in words from the list in the given 8×8 board:
HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US

- Variables: $R_1, C_3, C_5, R_8, \ldots,$
- Domains: $R_1 \in \{HOSES, LASER, SHEET, SNAIL, STEER\}, C_3 \in \{ALSO, SAME, \ldots\}$
- Constraints: $R_1[3] = C_3[1], \ldots$

Variant of crossword puzzle (practice problem)



Pack the following words in the given 8×8 board:
ZERO, ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE, TEN

Rules:
- All words must read either across or down, as in a crossword puzzle.
- No letters are adjacent unless they belong to one of the given words.
- The words are rookwise connected.
- Words overlap only when one is vertical and the other is horizontal.

Solution overview

- **CSP graph creation**
  - **Create a** *node* **for** *every variable*. **All possible** *domain values* **are initially** *assigned* **to the variable**
  - **Draw** *edges* **between nodes if there is a** *binary Constraint*. **Otherwise draw a** *hyper-edge* **between nodes with constraints involving more than two variables**
- **Constraint propagation**
  - **Reduce the** *valid domains* **of each** *variable* **by applying node consistency, arc / edge Consistency, K-Consistency, till no further reduction is possible. If a solution is found or the problem found to have no consistent solution, then terminate**
- **Search for solution**
  - **Apply** *search algorithms* **to find solutions**
  - **There are interesting properties of CSP graphs which lead of efficient algorithms in some cases:** *Trees, Perfect Graphs, Interval Graphs, etc.*
  - **Issues for Search: Backtracking Scheme, Ordering of Children, Forward Checking (Look-Ahead) using Dynamic Constraint Propagation**
  - **Solving by converting to** *satisfiability (SAT)* **problems**

Search formulation of CSP

- **Standard** ***search*** **formulation of CSP**
  - **Initial state: all unassigned variables**
  - **State: partial assignment of the variables**
  - **Successor function: assign a value to unassigned variables**
  - **Goal state: all variables are assigned and satisfies all constraints**
  - **Path cost: uniform path cost**

Constraint propagation

- **Constraints**
  - **Unary constraints or node constraints (eg. $x_i \neq 9$)**
  - **Binary constraints or edge between nodes (eg. $x_i \neq x_j$)**
  - **Higher order or hyper-edge between nodes (eg. $x_1 + x_2 = x_3$)**
- **Node consistency**
  - **For every variable V_i, remove all elements of D_i that do not satisfy the unary constraints for the variable**
  - **First step is to reduce the domains using node consistency**
- **Arc consistency**
  - **For every element x_{ij} of D_i, for every edge from V_i to V_j, remove x_{ij} if it has no consistent value(s) in other domains satisfying the Constraints**
  - **Continue to iterate using arc consistency till no further reduction happens.**
- **Path consistency**
  - **For every element y_{ij} of D_i, choose a path of length L with L variables, use a consistency checking method similar to above to reduce domains if possible**

Arc consistency check (AC-3)

AC-3(*csp*) // inputs - CSP with variables, domains, constraints

queue ⟵ local variable initialized to all arcs in csp
while *queue* is not empty **do**
(X_i, X_j) ⟵ pop(queue)
if Revise(*csp*, X_i, X_j) **then**
if size of D_i = 0 **then return** *false*
for each X_k **in** X_i.neighbors-{X_j} **do**
add (X_k, X_i) to *queue*
return *true*

Revise(*csp*, X_i, X_j)

revised ⟵ *false*
for each x **in** D_i **do**
if no value y in D_j allows (x, y) to satisfy constraint between X_i and X_j **then**
delete x from D_i
revised ⟵ *true*
return *revised*

Arc consistency check (AC-3)

```
AC-3(csp) // inputs - CSP with variables, domains, constraints
    queue ← local variable initialized to all arcs in csp
    while queue is not empty do
        (X_i, X_j) ← pop(queue)
        if Revise(csp, X_i, X_j) then
            if size of D_i = 0 then return false
            for each X_k in X_i.neighbors-{X_j} do
                add (X_k, X_i) to queue
    return true
```

```
Revise(csp, X_i, X_j)
    revised ← false
    for each x in D_i do
        if no value y in D_j allows (x, y) to satisfy constraint between X_i and X_j then
            delete x from D_i
            revised ← true
    return revised
```

Complexity?

AC-3 example

- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B$, $C < B$, $C < D$

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- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B, C < B, C < D$

queue: AB, BA, BC, CB, CD, DC

AC-3 example

- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B$, $C < B$, $C < D$

queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB

AC-3 example

- Variables: A, B, C, D
- Domain: {1, 2, 3}
- Constraints: $A \neq B$, $C < B$, $C < D$

queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB
No change in queue. queue=BA, BC, CB, CD, DC

AC-3 example

- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B$, $C < B$, $C < D$

queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB
No change in queue. queue=BA, BC, CB, CD, DC
pop(queue) // BA

AC-3 example

- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B$, $C < B$, $C < D$

queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB
No change in queue. queue=BA, BC, CB, CD, DC
pop(queue) // BA
No change in queue. queue=BC, CB, CD, DC

AC-3 example

- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B$, $C < B$, $C < D$

queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB
No change in queue. queue=BA, BC, CB, CD, DC
pop(queue) // BA
No change in queue. queue=BC, CB, CD, DC
pop(queue) // BC

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- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
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queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB
No change in queue. queue=BA, BC, CB, CD, DC
pop(queue) // BA
No change in queue. queue=BC, CB, CD, DC
pop(queue) // BC
Remove 1. $D_B = \{2, 3\}$

AC-3 example

- Variables: A, B, C, D
- Domain: $\{1, 2, 3\}$
- Constraints: $A \neq B$, $C < B$, $C < D$

queue: AB, BA, BC, CB, CD, DC
pop(queue) // AB
No change in queue. queue=BA, BC, CB, CD, DC
pop(queue) // BA
No change in queue. queue=BC, CB, CD, DC
pop(queue) // BC
Remove 1. $D_B = \{2, 3\}$
Add AB to queue. queue=CB, CD, DC, AB
pop(queue) // CB
Remove 3. $D_C = \{1, 2\}$
No change in queue. queue=CD, DC, AB
pop(queue) // CD
No change. queue=DC, AB
pop(queue) // DC
Remove 1. $D_D = \{2, 3\}$
No change. queue=AB
pop(queue) // AB
No change in queue. queue=$\emptyset$

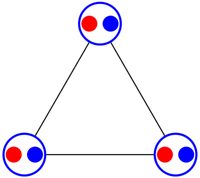
$A = \{1, 2, 3\}, B = \{2, 3\},$
$C = \{1, 2\}, D = \{2, 3\}.$

Sudoku

		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
7					x			8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1	y	3		

AC-3 limitations

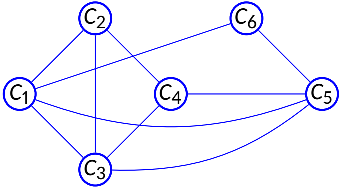
- After successful run of AC-3
  - There can be only one solution
  - There can be more than one solutions
  - There may be no solution and it fails to identify



Examination schedule

Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6

Is it possible to conduct all these exams in 3 days assuming one exam per day?



- **How does naive BFS & DFS perform?**

Backtracking search

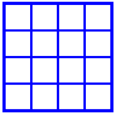
- **Backtracking is a basic search methodology for solving CSP**
- **Basic steps:**
  - **Assign one variable at a time**
    - **Fix ordering of variables (eg.**$C_1 = 1, C_2 = 3$ **is same as** $C_2 = 3, C_1 = 1$**)**
  - **Check constraint**
    - **Check with previously assigned variables**

Backtracking search

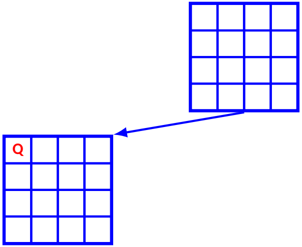
Backtrack(*assignment*)
    if *assignment* is complete **then return** *success*, *assignment*
    var ← Choose-unassigned-variable()
    for each *value* of Domain(*var*) **do**
        if *value* is consistent with the *assignment* **then**
            add *var* = *value* to *assignment*
            result = Backtrack(*assignment*)
            if *result* ≠ *failure* **return** *result*, *assignment*
    return *failure*

- **Choices:**
    - **Variable to be assigned next**
    - **Value to be assigned to the variable next**
    - **Early detection of failure**

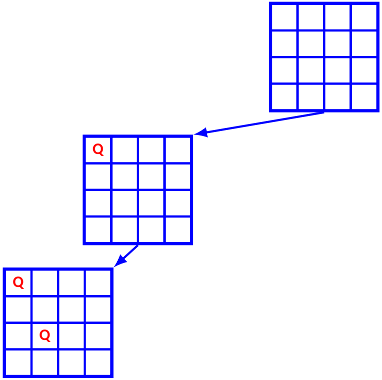
4 Queens



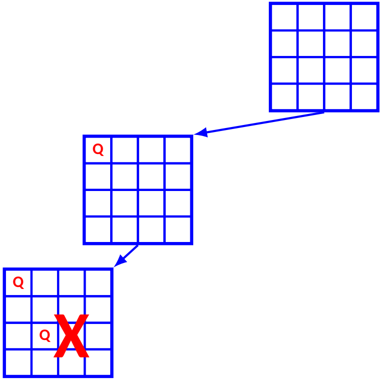
4 Queens



4 Queens



4 Queens



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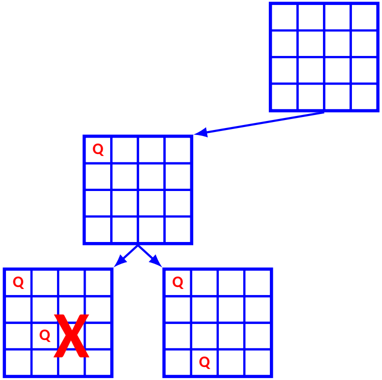

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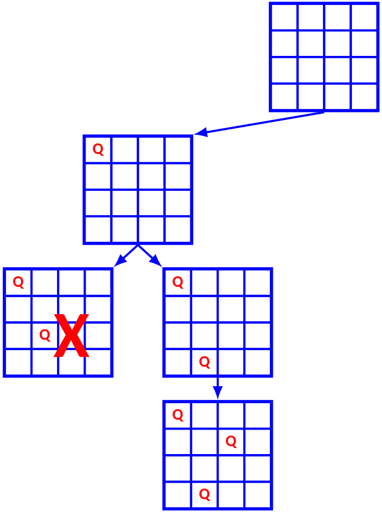

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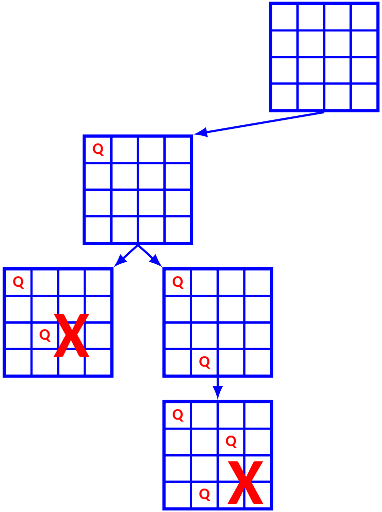
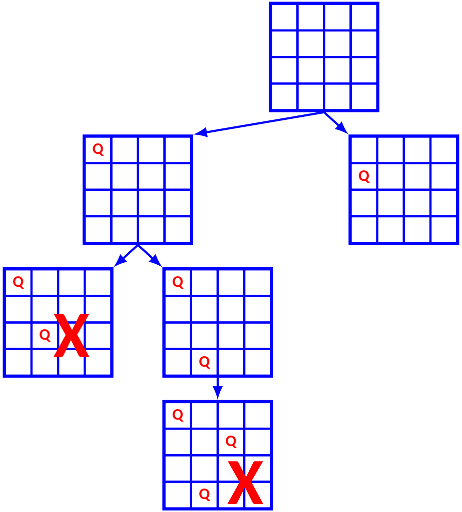
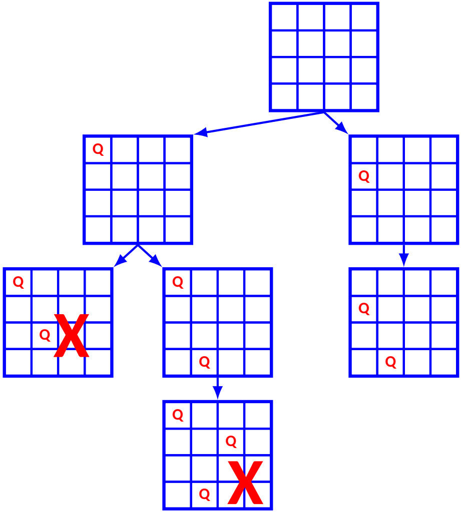


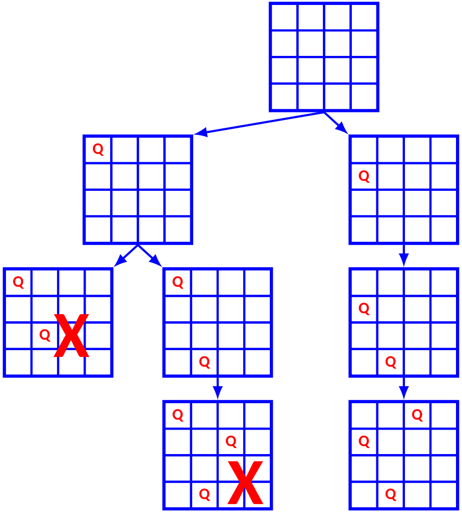
4 Queens



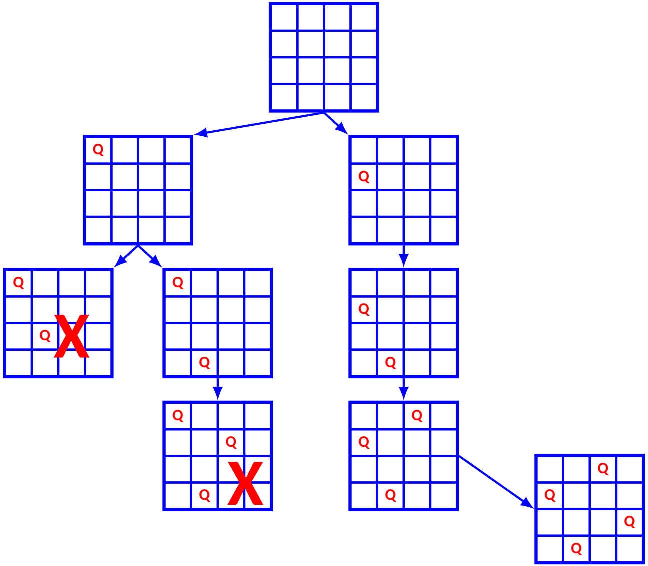
4 Queens



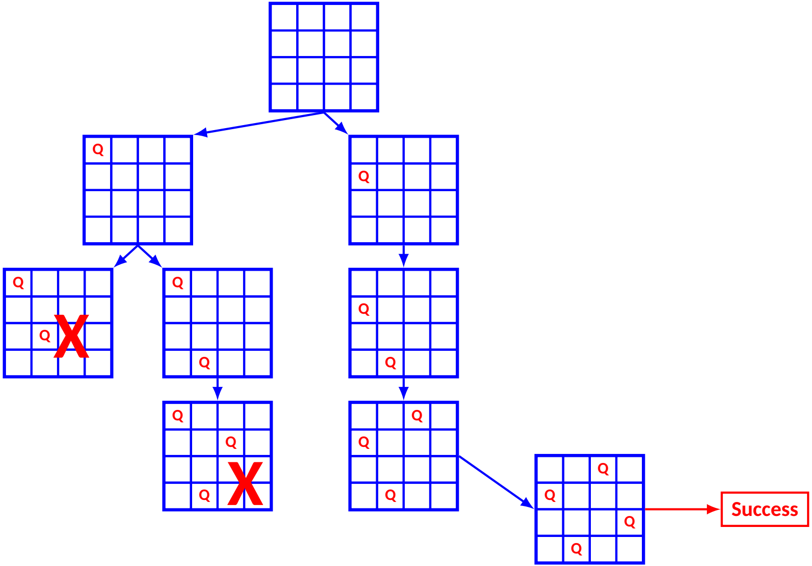
4 Queens



4 Queens



4 Queens

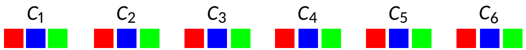


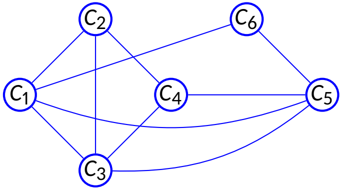
Heuristic strategy

- Variable ordering
  - Static or random
  - Minimum remaining values
    - Variable with fewest legal values (also known as most constrained variable)
  - Degree heuristic
    - Variable with the largest number of constraints on other unassigned variables
- Choice of value
  - Least constraining value
    - Value that leaves most choices for the neighboring variables in the constraint graph

Forward checking

- **Forward checking propagates information from assigned to unassigned variables**





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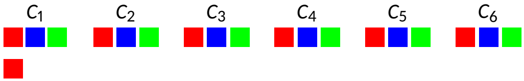

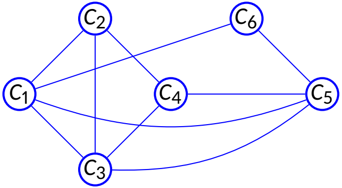

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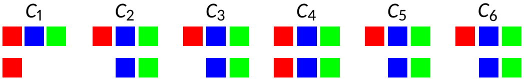

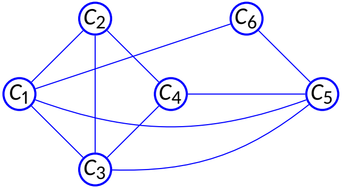

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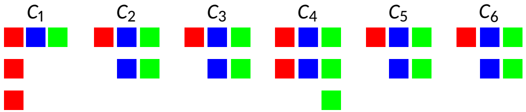

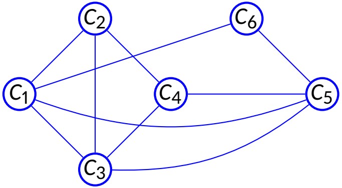

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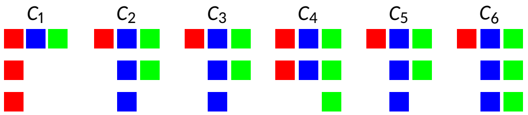
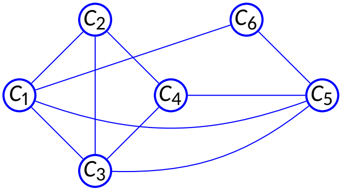
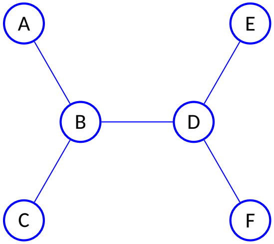




Special cases

- **General CSP problem is NP-Complete**
- **For** *perfect graphs*, *chordal graphs*, *interval graphs*, **the graph coloring problem can be solved in polynomial time**
- **Tree structured CSP can be solved in polynomial time**



Thank you!