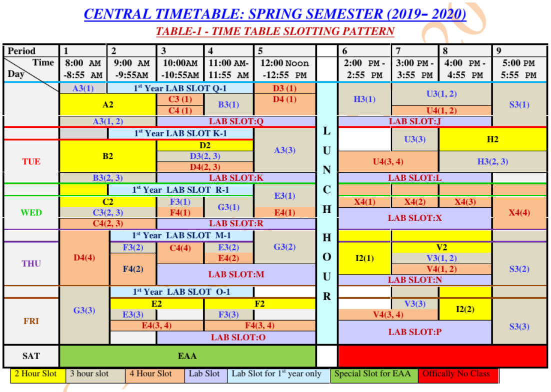
Artificial Intelligence: Foundations & Applications

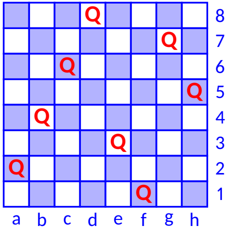
Introduction to Constraint Satisfaction Problem

Prof. Partha P. Chakrabarti & Arijit Mondal
Indian Institute of Technology Kharagpur

Examples of CSP

- Crossword puzzle
- N-queens on chess board
- Knapsack
- Assembly scheduling
- Operations research
- Map coloring
- Time tabling
- Airline/train scheduling
- Cryptic puzzle
- Boolean satisfiability
- Car sequencing
- Scene labeling
- etc.





CSP formulation

- **Variables**
  - **A set of *decision* variables** $x_1, x_2, \ldots, x_n$

CSP formulation

- **Variables**
  - **A set of *decision* variables** $x_1, x_2, \ldots, x_n$
- **Domain of variables**
  - **Each variable has a domain (discrete or continuous)** $D_1, D_2, \ldots, D_n$ **from which it can take a value.**

CSP formulation

- **Variables**
  - **A set of *decision* variables** $x_1, x_2, \ldots, x_n$
- **Domain of variables**
  - **Each variable has a domain (discrete or continuous)** $D_1, D_2, \ldots, D_n$ **from which it can take a value.**
- **Satisfaction constraint**
  - **A finite set of satisfaction constraints** $C_1, C_2, \ldots, C_m$
  - **A constraint can be unary, binary or among many variables. Given a value of variables, any constraint will yield *yes* or *no* only**

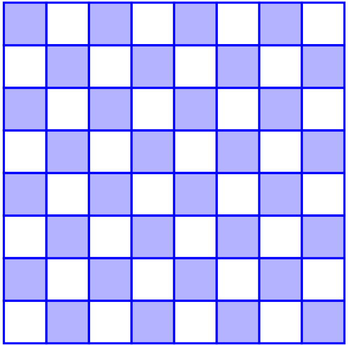
CSP formulation

- **Variables**
  - **A set of *decision* variables** $x_1, x_2, \ldots, x_n$
- **Domain of variables**
  - **Each variable has a domain (discrete or continuous)** $D_1, D_2, \ldots, D_n$ **from which it can take a value.**
- **Satisfaction constraint**
  - **A finite set of satisfaction constraints** $C_1, C_2, \ldots, C_m$
  - **A constraint can be unary, binary or among many variables. Given a value of variables, any constraint will yield *yes* or *no* only**
- **Cost function for optimization (optional)**
  - **A set of optimization functions (typically** $\min, \max$**)** $O_1, O_2, \ldots, O_p$

CSP formulation

- Variables
  - A set of *decision* variables $x_1, x_2, \ldots, x_n$
- Domain of variables
  - Each variable has a domain (discrete or continuous) $D_1, D_2, \ldots, D_n$ from which it can take a value.
- Satisfaction constraint
  - A finite set of satisfaction constraints $C_1, C_2, \ldots, C_m$
  - A constraint can be unary, binary or among many variables. Given a value of variables, any constraint will yield *yes* or *no* only
- Cost function for optimization (optional)
  - A set of optimization functions (typically $\min, \max$) $O_1, O_2, \ldots, O_p$
- Solution
  - A consistent assignment of domain values to each variable so that all constraints are satisfied and the optimization criteria (if any) are met.

N-Queens

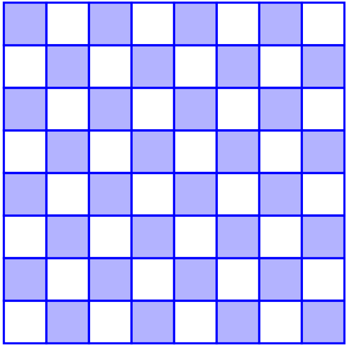


Need to place N-queens on this board

Rules:

- No queens are attacking each other

N-Queens



Need to place N-queens on this board

Rules:

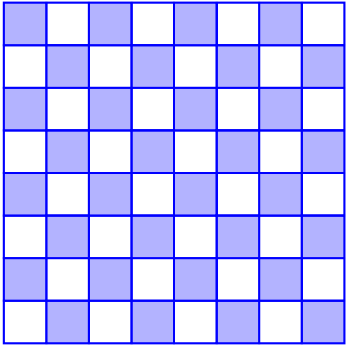
- No queens are attacking each other

- Variables: x_{ij} - queen is in cell (i, j),
- Domains: $D_{ij} \in \{0, 1\}$
- Constraints: $\sum_i x_{ij} = 1, \sum_j x_{ij} = 1, \sum_{i,j} x_{ij} = N,$

  $x_{ij} + x_{(i+k)(j+k)} \leq 1, \quad x_{ij} + x_{(i+k)(j-k)} \leq 1,$

  k is in appropriate range
- Search space $2^{64} = 18, 446, 744, 073, 709, 551, 616$

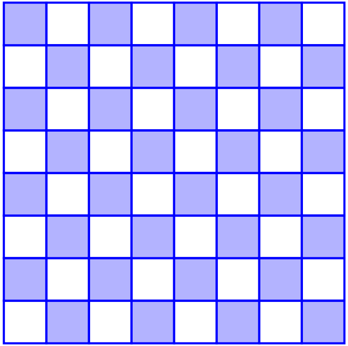
N-Queens (alternative model)



Need to place N-queens on this board

Rules:
- No queens are attacking each other

N-Queens (alternative model)

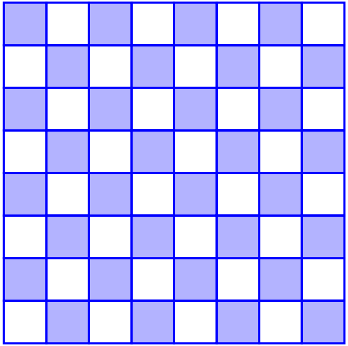


Need to place N-queens on this board

Rules:
- No queens are attacking each other

- Variables: x_i
- Domains: $D_i \in \{1, 2, \ldots, 8\}$
- Constraints: $\ldots$
- Search space $8^8 = 16,777,216$

N-Queens (alternative model)



Need to place N-queens on this board

Rules:
- No queens are attacking each other

- Variables: x_i
- Domains: $D_i \in \{1, 2, \ldots, 8\}$
- Constraints: $\ldots$
- Search space $8^8 = 16,777,216$

Other variants:
- At least a queen on the main diagonal
- Two queens on the two main diagonals
- Enumeration of all solutions

Examination schedule

Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6

Examination schedule

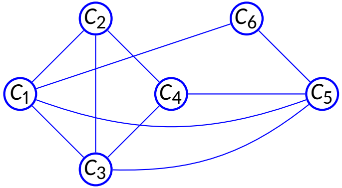
Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6

Is it possible to conduct all these exams in 3 days assuming one exam per day?

Examination schedule

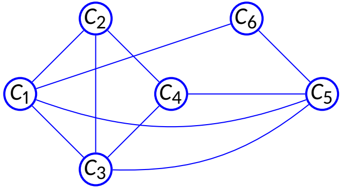
Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6

Is it possible to conduct all these exams in 3 days assuming one exam per day?



Examination schedule

Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6

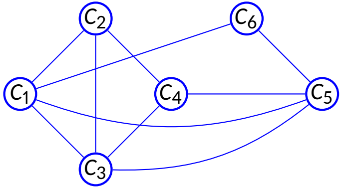


Is it possible to conduct all these exams in 3 days assuming one exam per day?

- Variables: x_i - slot for subject C_i
- Domains: $D_i \in \{1, 2, 3\}$
- Constraints: $x_1 \neq x_2, x_1 \neq x_3, \ldots$

Examination schedule

Student	Subjects
S_1	C_1, C_2, C_3
S_2	C_2, C_3, C_4
S_3	C_3, C_4
S_4	C_3, C_4, C_5
S_5	C_1, C_5, C_6



Is it possible to conduct all these exams in 3 days assuming one exam per day?

- Variables: x_i - slot for subject C_i
- Domains: $D_i \in \{1, 2, 3\}$
- Constraints: $x_1 \neq x_2, x_1 \neq x_3, \ldots$

Graph coloring problem.

Airport gate scheduling

Flight	Arrv. time	Dept. time
F1	0715	0815
F2	0800	0900
F3	0830	0930
F4	0845	0945
F5	0915	1015
F6	0845	0945

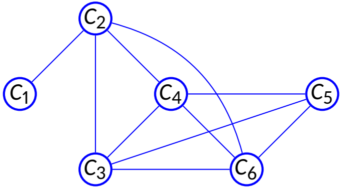
Airport gate scheduling

Flight	Arrv. time	Dept. time
F1	0715	0815
F2	0800	0900
F3	0830	0930
F4	0845	0945
F5	0915	1015
F6	0845	0945

Is it possible to schedule all flights using 3 gates?

Airport gate scheduling

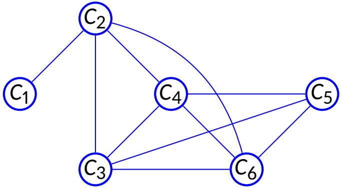
Flight	Arrv. time	Dept. time
F1	0715	0815
F2	0800	0900
F3	0830	0930
F4	0845	0945
F5	0915	1015
F6	0845	0945



Is it possible to schedule all flights using 3 gates?

Airport gate scheduling

Flight	Arrv. time	Dept. time
F1	0715	0815
F2	0800	0900
F3	0830	0930
F4	0845	0945
F5	0915	1015
F6	0845	0945

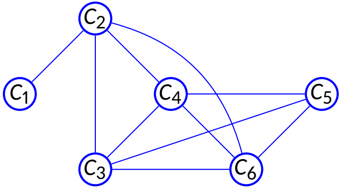


Is it possible to schedule all flights using 3 gates?

- Variables: x_i - slot for subject C_i
- Domains: $D_i \in \{1, 2, 3\}$
- Constraints: $x_1 \neq x_2, x_1 \neq x_3, \ldots$

Airport gate scheduling

Flight	Arrv. time	Dept. time
F1	0715	0815
F2	0800	0900
F3	0830	0930
F4	0845	0945
F5	0915	1015
F6	0845	0945



Is it possible to schedule all flights using 3 gates?

- Variables: x_i - slot for subject C_i
- Domains: $D_i \in \{1, 2, 3\}$
- Constraints: $x_1 \neq x_2, x_1 \neq x_3, \ldots$

Interval Graphs.

Cryptarithmetic

		S	E	N	D
+		M	O	R	E
	M	O	N	E	Y

Cryptarithmetic

	S	E	N	D
+	M	O	R	E
M	O	N	E	Y

- Variables: $S, E, N, D, M, O, R, Y,$
- Domains: $D_i \in \{0, 1, \ldots, 9\}$
- Constraints: All different, $10 \times M + O = S + M + C_{1000}, \ldots$

Cryptarithmetic

	S	E	N	D
+	M	O	R	E
M	O	N	E	Y

- Variables: $S, E, N, D, M, O, R, Y,$
- Domains: $D_i \in \{0, 1, \ldots, 9\}$
- Constraints: All different, $10 \times M + O = S + M + C_{1000}, \ldots$

MiniZinc implementation:

```
include "alldifferent.mzn";

var 1..9: S; var 0..9: E; var 0..9: N; var 0..9: D;
var 1..9: M; var 0..9: O; var 0..9: R; var 0..9: Y;

constraint
              1000 * S + 100 * E + 10 * N + D
            + 1000 * M + 100 * O + 10 * R + E
= 10000 * M + 1000 * O + 100 * N + 10 * E + Y;

constraint alldifferent([S,E,N,D,M,O,R,Y]);

solve satisfy;
```

Knapsack

- **There are n items namely, $O_1, O_2, \ldots, O_n$. Item O_i weighs w_i and provides profit of p_i. Target is to select a subset of the items such that the total weight of the items does not exceed W and profit is maximized.**

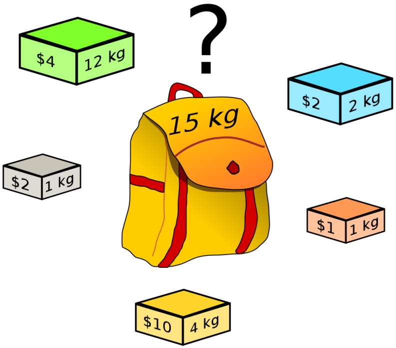


image source: Wikipedia

Knapsack

- **There are n items namely, $O_1, O_2, \ldots, O_n$. Item O_i weighs w_i and provides profit of p_i. Target is to select a subset of the items such that the total weight of the items does not exceed W and profit is maximized.**

  - **Variables:** x_i **- selection of** i**th item**
  - **Domains:** $\{0, 1\}$
  - **Constraints:** $\sum_i x_i \times w_i \leq W$
  - **Optimization function:** $\sum_i x_i \times p_i$

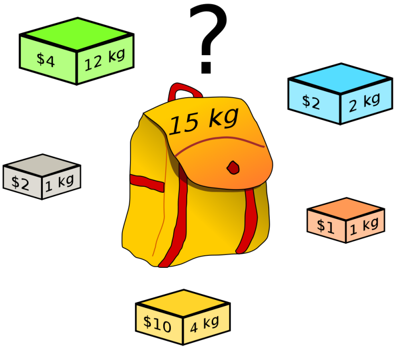


image source: Wikipedia

Warehouse planning

- **There are** n **possible locations to setup warehouses** (W) **which will deliver goods to** m **customers** (C)**. Cost to setup** W_j **warehouse is** f_j**. Customer** C_i **has a demand of** d_i **which needs to fulfilled by the warehouses. Delivery cost per unit item from** W_j **to** C_i **is** c_{ji}**. Target is to minimize total cost to serve the required demands.**

Warehouse planning

- **There are n possible locations to setup warehouses (W) which will deliver goods to m customers (C). Cost to setup W_j warehouse is f_j. Customer C_i has a demand of d_i which needs to fulfilled by the warehouses. Delivery cost per unit item from W_j to C_i is c_{ji}. Target is to minimize total cost to serve the required demands.**
  - **Variables:** x_j **- warehouse location,** y_{ji} **- amount served by** W_j **to** C_i
  - **Domains:** $x_j \in \{0, 1\}$, $y_{ji} \in \{0, \infty\}$
  - **Constraints:** $\sum_j y_{ji} = d_i$,

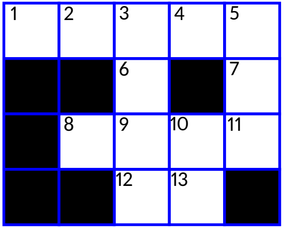
Warehouse planning

- **There are n possible locations to setup warehouses (W) which will deliver goods to m customers (C). Cost to setup W_j warehouse is f_j. Customer C_i has a demand of d_i which needs to fulfilled by the warehouses. Delivery cost per unit item from W_j to C_i is c_{ji}. Target is to minimize total cost to serve the required demands.**
  - **Variables:** x_j **- warehouse location,** y_{ji} **- amount served by** W_j **to** C_i
  - **Domains:** $x_j \in \{0, 1\}$, $y_{ji} \in \{0, \infty\}$
  - **Constraints:** $\sum_j y_{ji} = d_i$, $\sum_i y_{ji} - x_j \left(\sum_i d_i \right) \leq 0$

Warehouse planning

- **There are n possible locations to setup warehouses (W) which will deliver goods to m customers (C). Cost to setup W_j warehouse is f_j. Customer C_i has a demand of d_i which needs to fulfilled by the warehouses. Delivery cost per unit item from W_j to C_i is c_{ji}. Target is to minimize total cost to serve the required demands.**
  - **Variables:** x_j **- warehouse location,** y_{ji} **- amount served by** W_j **to** C_i
  - **Domains:** $x_j \in \{0, 1\}$, $y_{ji} \in \{0, \infty\}$
  - **Constraints:** $\sum_j y_{ji} = d_i$, $\sum_i y_{ji} - x_j \left(\sum_i d_i \right) \leq 0$
  - **Optimization function:** $\sum_i x_j \times f_j + \sum_{i,j} c_{ji} \times y_{ji}$

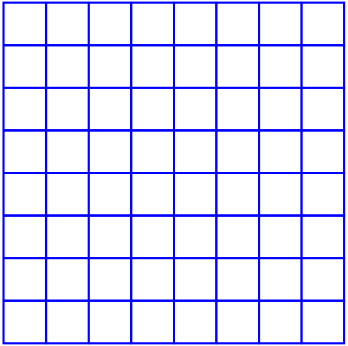
Crossword puzzle



Fill in words from the list in the given 8×8 board:
HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US

- Variables: $R_1, C_3, C_5, R_8, \ldots,$
- Domains: $R_1 \in \{HOSES, LASER, SHEET, SNAIL, STEER\}, C_3 \in \{ALSO, SAME, \ldots\}$
- Constraints: $R_1[3] = C_3[1], \ldots$

Variant of crossword puzzle (practice problem)



Pack the following words in the given 8 × 8 board:
ZERO, ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE, TEN

Rules:

- All words must read either across or down, as in a crossword puzzle.
- No letters are adjacent unless they belong to one of the given words.
- The words are rookwise connected.
- Words overlap only when one is vertical and the other is horizontal.

Thank you!