Artificial Intelligence: Foundations & Applications

Introduction to Probabilistic Reasoning

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Need for probabilistic model

- Till now, we have considered that intelligent agent has
  - Known environment
  - Full observability
  - Deterministic world
- Reasons for using probability
  - Example: $Toothache \implies Cavity$
    - Toothache can be caused by many other ways also, eg.
      $Toothache \implies GumDisease \lor Cavity \lor WisdomTeeth \lor \ldots$
  - Specifications become too large
  - Theoretical ignorance
  - Practical ignorance

Probabilistic reasoning

- Useful for prediction
  - Analyzing causal effect, predicting outcome
    - Given that I have cavity, what is the chance that I will have toothache?
- Useful for diagnosis
  - Analyzing causal effect, finding out reasons for a given effect
    - Given that I have toothache, what is the chance that it is caused by a cavity?
- Require a methodology to analyze both the scenarios

Axioms of probability

- **All probabilities lie between 0 and 1, ie.,** $0 \leq P(A) \leq 1$
- $P(True) = 1$ **and** $P(False) = 0$
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

- $P(A \wedge B) = P(A|B) \times P(B) = P(B|A) \times P(A)$
  - $P(A) = \dfrac{P(A|B) \times P(B)}{P(B|A)}$
  - **Bayes' rule**

Joint probability

Color \ Type	EV	SUV	Sedan	Truck
Red	0.05	0.20	0.00	0.10
Green	0.10	0.00	0.10	0.00
Black	0.00	0.10	0.05	0.10
White	0.10	0.00	0.10	0.00

- **What is the probability of a vehicle to be EV and Red?**

Joint probability

Color \ Type	EV	SUV	Sedan	Truck
Red	0.05	0.20	0.00	0.10
Green	0.10	0.00	0.10	0.00
Black	0.00	0.10	0.05	0.10
White	0.10	0.00	0.10	0.00

- **What is the probability of a vehicle to be EV and Red?**
- **What is the probability of a Red vehicle?**

Joint probability

Color \ Type	EV	SUV	Sedan	Truck
Red	0.05	0.20	0.00	0.10
Green	0.10	0.00	0.10	0.00
Black	0.00	0.10	0.05	0.10
White	0.10	0.00	0.10	0.00

- **What is the probability of a vehicle to be EV and Red?**
- **What is the probability of a Red vehicle?**
  - **Marginal probability:** $\sum_{T} P(C = Red \wedge T = *)$

Joint probability

Type / Color	EV	SUV	Sedan	Truck
Red	0.05	0.20	0.00	0.10
Green	0.10	0.00	0.10	0.00
Black	0.00	0.10	0.05	0.10
White	0.10	0.00	0.10	0.00

- **What is the probability of a vehicle to be EV and Red?**
- **What is the probability of a Red vehicle?**
  - **Marginal probability:** $\sum_{T} P(C = Red \wedge T = *)$
- **What is the probability of a vehicle to be EV given that it is Red?**

Joint probability

Color \ Type	EV	SUV	Sedan	Truck
Red	0.05	0.20	0.00	0.10
Green	0.10	0.00	0.10	0.00
Black	0.00	0.10	0.05	0.10
White	0.10	0.00	0.10	0.00

- **What is the probability of a vehicle to be EV and Red?**
- **What is the probability of a Red vehicle?**
  - **Marginal probability:** $\sum_{T} P(C = Red \wedge T = *)$
- **What is the probability of a vehicle to be EV given that it is Red?**
  - **Conditional probability:** $\frac{P(C = Red \wedge T = EV)}{P(C = Red)}$

Independence

- Two variables are independent if $P(X, Y) = P(X) \times P(Y)$ holds
  - It means that their joint distribution factors into a product two distributions
  - This can be expressed as $P(X|Y) = P(X)$
- Independence is a simplifying modeling assumption
  - Empirical joint distributions can be at best close to independent

Example: Independence

T	W	P(T,W)
Hot	Sun	0.4
Hot	Rain	0.1
Cold	Sun	0.2
Cold	Rain	0.3

Example: Independence

T	W	P(T,W)
Hot	Sun	0.4
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- **Are *T* and *W* independent?**

Example: Independence

T	W	P(T,W)
Hot	Sun	0.4
Hot	Rain	0.1
Cold	Sun	0.2
Cold	Rain	0.3

- **Are *T* and *W* independent?**
- **Find marginal probabilities**

Example: Independence

T	W	P(T,W)
Hot	Sun	0.4
Hot	Rain	0.1
Cold	Sun	0.2
Cold	Rain	0.3

- **Are T and W independent?**
- **Find marginal probabilities**
  - $P(T = Hot) = ?$, $P(T = Cold) = ?$
  - $P(W = Sun) = ?$, $P(W = Rain) = ?$

Example: Independence

T	W	P(T,W)
Hot	Sun	0.4
Hot	Rain	0.1
Cold	Sun	0.2
Cold	Rain	0.3

- **Are T and W independent?**
- **Find marginal probabilities**
  - $P(T = Hot) = ?$, $P(T = Cold) = ?$
  - $P(W = Sun) = ?$, $P(W = Rain) = ?$
- **Now check for independence**

Example: Independence

- **Tossing of N fair coins**

$P(X_1)$	
H	**0.5**
T	**0.5**

$P(X_2)$	
H	**0.5**
T	**0.5**

$\ldots\ldots$

$P(X_n)$	
H	**0.5**
T	**0.5**

$P(X_1, X_2, \ldots, X_n)$

X_1	X_2	X_3	$\ldots$	$\ldots$	X_n	P
H					**H**	
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	
T					**H**	

$\}\, 2^n$

Chain rule

- **Product rule**

$$P(X, Y) = P(X) \times P(Y|X)$$

- **Chain rule**

$$\begin{aligned} P(X_1, X_2, \ldots, X_n) &= P(X_1) \times P(X_2|X_1) \times P(X_3|X_1, X_2) \times \ldots \times P(X_N|X_1, \ldots, X_{n-1}) \\ &= \prod_i P(X_i|X_1, \ldots, X_{i-1})) \end{aligned}$$

Conditional independence

- $P(Toothache, Cavity, Catch)$
- **If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:**

  $P(+catch| + toothache, +cavity) = P(+catch| + cavity)$

- **The same independence holds if I don't have a cavity:**

  $P(+catch| + toothache, -cavity) = P(+catch| - cavity)$

- **Catch is conditionally independent of Toothache given Cavity:**

  $P(Catch|Toothache, Cavity) = P(Catch|Cavity)$

Conditional independence

- **Absolute independence is very rare**
- **Conditional independence is the most basic and robust form of knowledge about uncertain environments**
- **Mathematically it is defined as -** X **is conditionally independent of** Y **given** Z

  $$P(X, Y|Z) = P(X|Z) \times P(Y|Z)$$

- **It can also be shown that**

  $$P(X|Z, Y) = P(X|Z)$$

Belief network

- Qualitative information
  - A belief network is a graph consist of the following:
  - Nodes – Set of random variables
  - Edges – Dependency of nodes. $X \longrightarrow Y$ means X has direct influence on Y
- Quantitative information
  - Each node has conditional probability table that quantifies the effects the parents have on the node

- The network is a directed acyclic graph (DAG), ie., without any cycle

Example

- **Consider the following knowledge base**
  - *Rain* **- Raining**
  - *Traffic* **- There may be traffic if it rains**
  - *Umbrella* **- People may carry umbrella if it rains**

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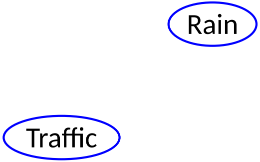

Example

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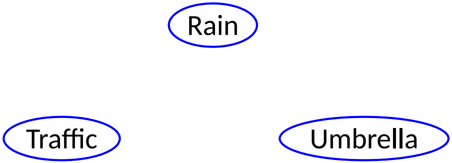

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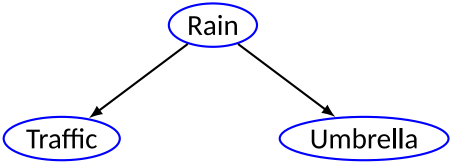
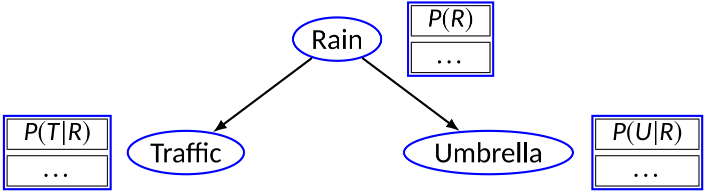
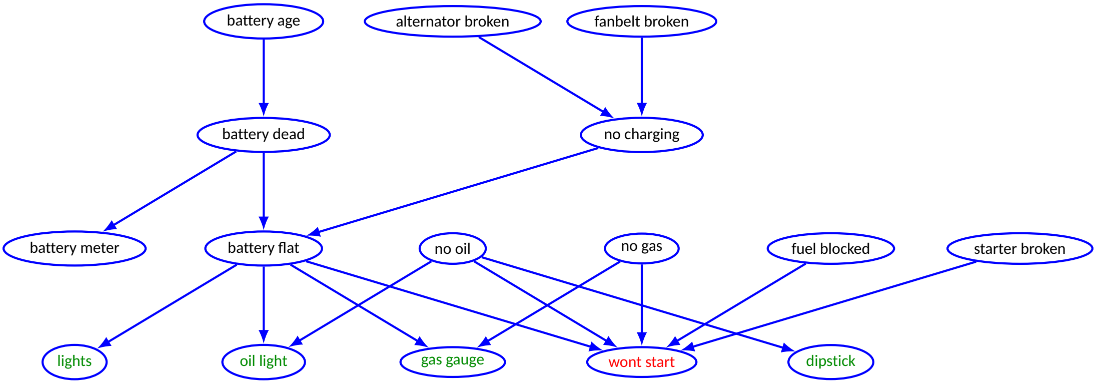


Example

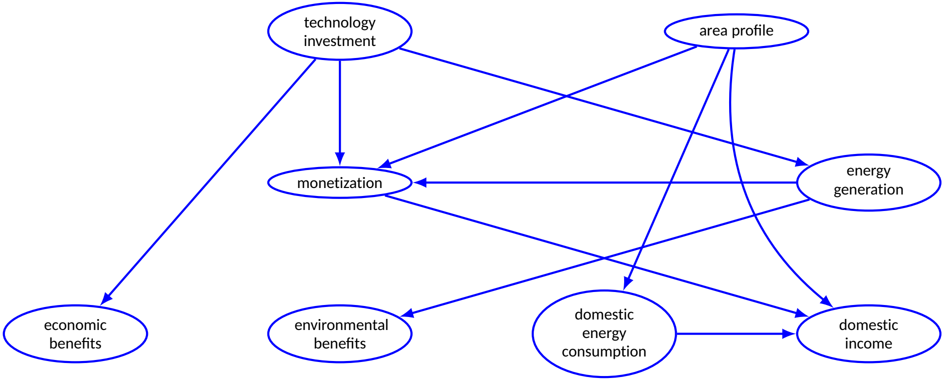
- **Consider the following knowledge base**
  - *Rain* - **Raining**
  - *Traffic* - **There may be traffic if it rains**
  - *Umbrella* - **People may carry umbrella if it rains**



Belief network example: car



Belief network example: renewable energy



Classical example

- Burglar alarm at a home
  - It works almost perfectly
  - Sometime alarm goes off if there is a earthquake
- John calls police when he hears the alarm but sometimes he misinterpret telephone ringing as alarm and calls police too.
- Mary also calls police. But she loves loud music, therefore, she misses the alarm sometime.

Belief network example

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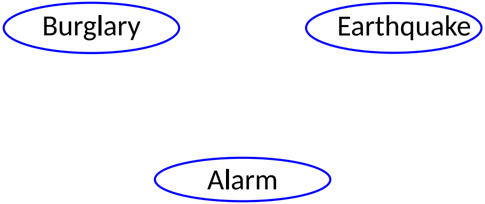
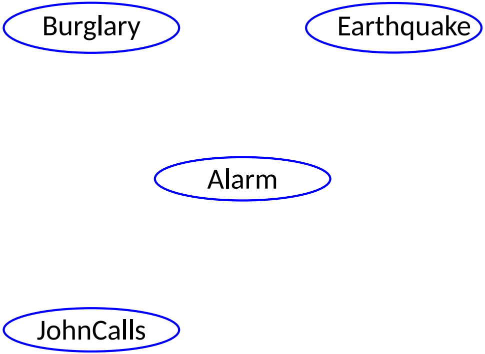
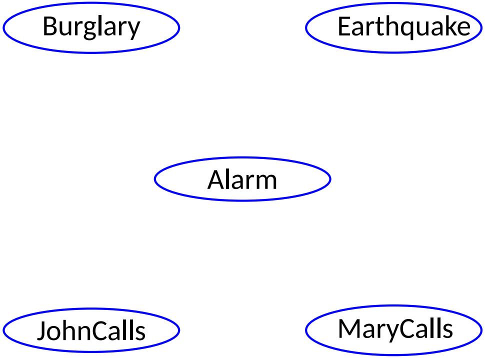
Belief network example



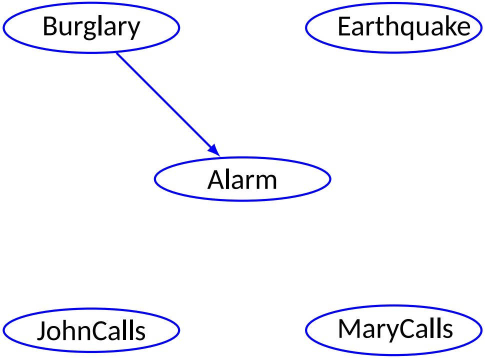
Belief network example



Belief network example



Belief network example



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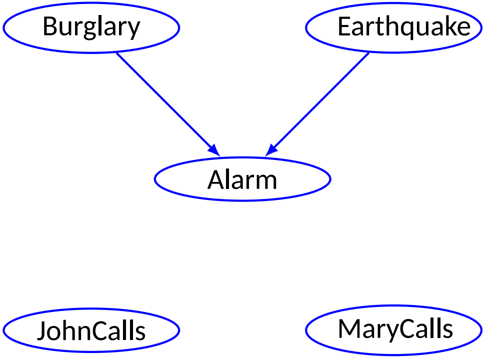
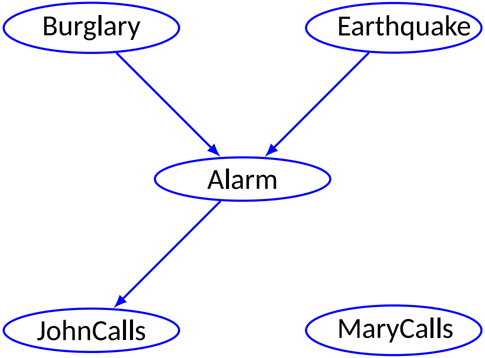


Belief network example



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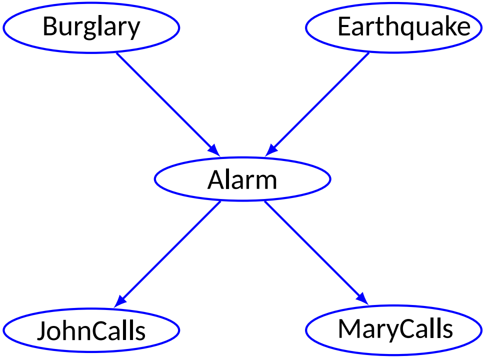
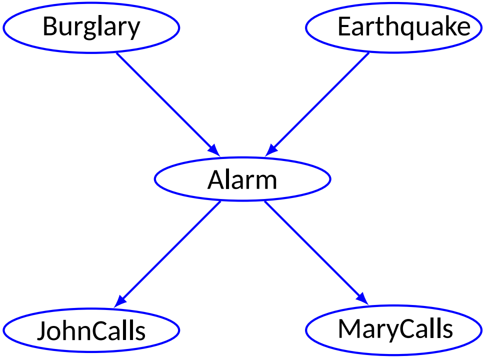


Belief network example

$P(B)$
0.001



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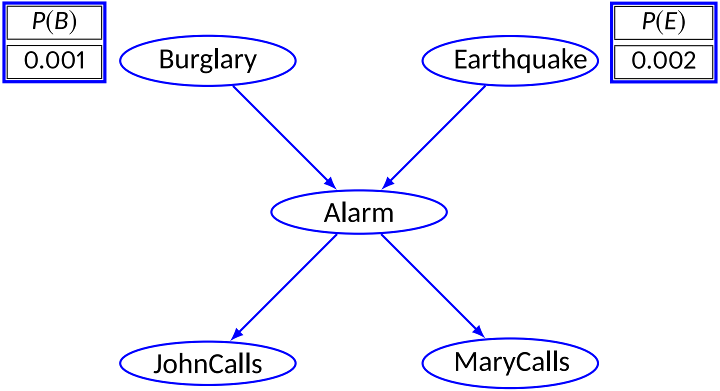
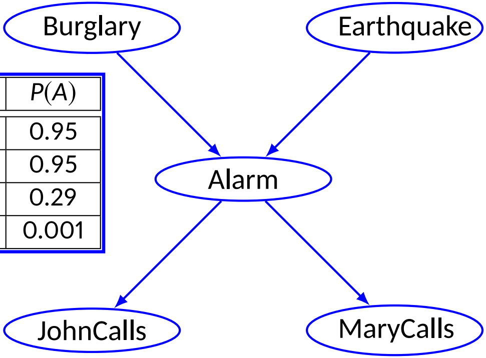


Belief network example

$P(B)$
0.001

$P(E)$
0.002

B	E	$P(A)$
T	T	0.95
T	F	0.95
F	T	0.29
F	F	0.001



Belief network example

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0.001

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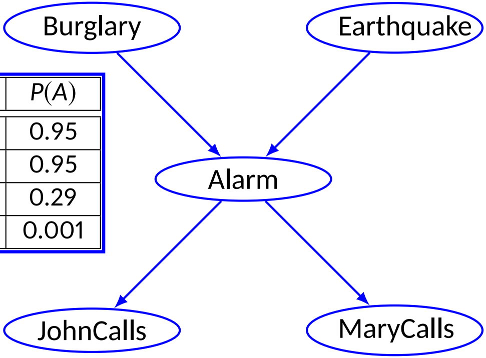


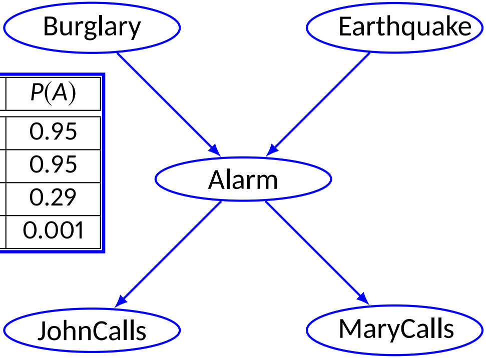
B	E	$P(A)$
T	T	0.95
T	F	0.95
F	T	0.29
F	F	0.001

A	$P(J)$
T	0.90
F	0.05

Belief network example

$P(B)$
0.001

$P(E)$
0.002



B	E	$P(A)$
T	T	0.95
T	F	0.95
F	T	0.29
F	F	0.001

A	$P(J)$
T	0.90
F	0.05

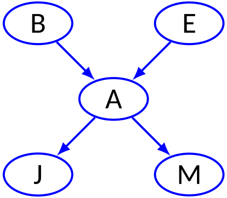
A	$P(M)$
T	0.70
F	0.01

Probabilities in Belief Nets

- **Bayes' net implicitly encode joint distributions**
  - **This can be determined from local conditional distributions**
  - **Need to multiply relevant conditional probabilities**
  - $P(X_1, X_2, \ldots, X_n) = \prod_{i}^{n} P(X_i | parents(X_i))$

Joint probability distribution

- **Probability of the event that the alarm has sounded but neither a burglary nor an earthquake has occurred, and both Mary and John call:**



$P(B)$
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0.002

B	E	$P(A)$
T	T	0.95
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A	$P(M)$
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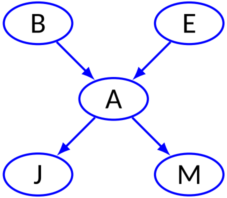
Joint probability distribution

- **Probability of the event that the alarm has sounded but neither a burglary nor an earthquake has occurred, and both Mary and John call:**

  $P(J \wedge M \wedge A \wedge \neg B \wedge \neg E)$



$P(B)$
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$P(E)$
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T	F	0.95
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F	F	0.001

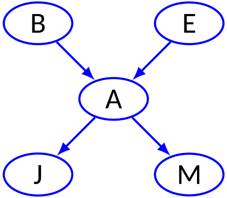
A	$P(J)$
T	0.90
F	0.05

A	$P(M)$
T	0.70
F	0.01

Joint probability distribution

- **Probability of the event that the alarm has sounded but neither a burglary nor an earthquake has occurred, and both Mary and John call:**

$$P(J \wedge M \wedge A \wedge \neg B \wedge \neg E)$$
$$= P(J|A) \times P(M|A) \times P(A|\neg B \wedge \neg E) \times P(\neg B) \times P(\neg E)$$
$$= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998$$
$$= 0.00062$$



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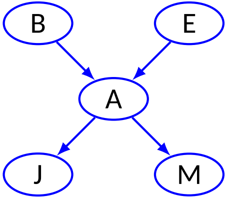
Joint probability distribution: $P(A)$

$P(A)$



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T	0.90
F	0.05

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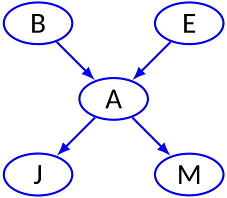
T	0.70
F	0.01

Joint probability distribution: $P(A)$

$$P(A) = P(A\bar{B}\bar{E}) + P(A\bar{B}E) + P(AB\bar{E}) + P(ABE)$$



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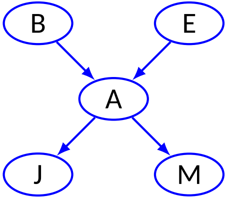
Joint probability distribution: $P(A)$

$$
\begin{aligned}
P(A) \\
&= P(A\bar{B}\bar{E}) + P(A\bar{B}E) + P(AB\bar{E}) + P(ABE) \\
&= P(A|\bar{B}\bar{E}) \times P(\bar{B}\bar{E}) + P(A|\bar{B}E) \times P(\bar{B}E) + P(A|B\bar{E}) \times P(B\bar{E}) + P(A|BE) \times P(BE)
\end{aligned}
$$



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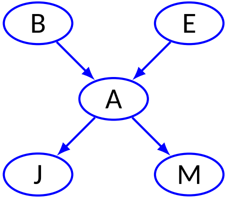
A	$P(M)$
T	0.70
F	0.01

Joint probability distribution: $P(A)$

$$
\begin{aligned}
P(A) \\
&= P(A\bar{B}\bar{E}) + P(A\bar{B}E) + P(AB\bar{E}) + P(ABE) \\
&= P(A|\bar{B}\bar{E}) \times P(\bar{B}\bar{E}) + P(A|\bar{B}E) \times P(\bar{B}E) + P(A|B\bar{E}) \times P(B\bar{E}) + P(A|BE) \times P(BE) \\
&= 0.001 \times 0.999 \times 0.998 + 0.29 \times 0.999 \times 0.002 + 0.95 \times 0.001 \times 0.998 \\
&\quad + 0.95 \times 0.001 \times 0.002 \\
&= 0.0025
\end{aligned}
$$



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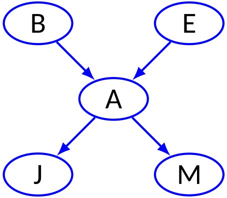
Joint probability distribution: $P(J)$

$P(J)$



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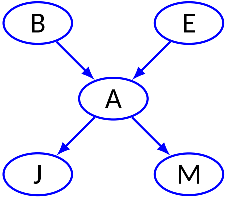
Joint probability distribution: $P(J)$

$P(J)$

$= P(JA) + P(J\bar{A})$



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A	$P(M)$
T	0.70
F	0.01

Joint probability distribution: $P(J)$

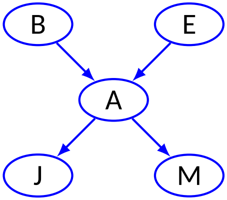
$P(J)$

$= P(JA) + P(J\bar{A})$

$= P(J|A) \times P(A) + P(J|\bar{A}) \times P(\bar{A})$

$= 0.9 \times 0.0025 + 0.05 \times (1 - 0.0025)$

$= 0.052125$



$P(B)$
0.001

$P(E)$
0.002

B	E	$P(A)$
T	T	0.95
T	F	0.95
F	T	0.29
F	F	0.001

A	$P(J)$
T	0.90
F	0.05

A	$P(M)$
T	0.70
F	0.01

Joint probability distribution: $P(J)$

$P(J)$

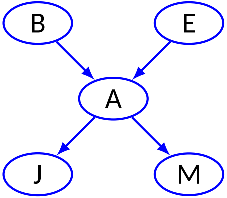
$= P(JA) + P(J\bar{A})$

$= P(J|A)\times P(A) + P(J|\bar{A})\times P(\bar{A})$

$= 0.9\times 0.0025 + 0.05\times(1 - 0.0025)$

$= 0.052125$

$P(AB)$



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T	F	0.95
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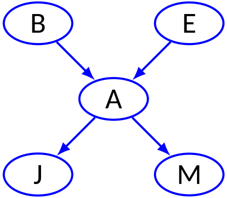
$= 0.052125$

$P(AB)$

$= P(ABE) + P(AB\bar{E})$



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$= 0.9 \times 0.0025 + 0.05 \times (1 - 0.0025)$

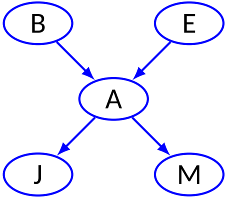
$= 0.052125$

$P(AB)$

$= P(ABE) + P(AB\bar{E})$

$= 0.95 \times 0.001 \times 0.002 + 0.95 \times 0.001 \times 0.998$

$= 0.00095$



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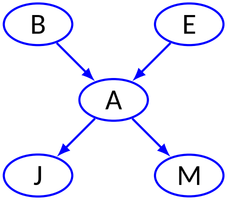
F	0.05

A	$P(M)$
T	0.70
F	0.01

Joint probability distribution: $P(JB)$

$P(JB)$



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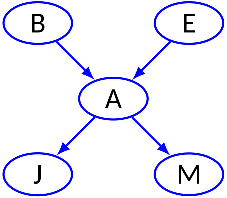
A	$P(M)$
T	0.70
F	0.01

Joint probability distribution: $P(JB)$

$P(JB)$

$= P(JBA) + P(JB\bar{A})$



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T	F	0.95
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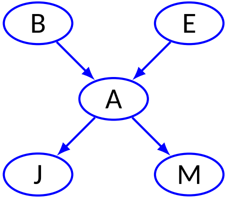
T	0.70
F	0.01

Joint probability distribution: $P(JB)$

$P(JB)$

$= P(JBA) + P(JB\bar{A})$

$= P(J|AB) \times P(AB) + P(J|\bar{A}B) \times P(\bar{A}B)$



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Joint probability distribution: $P(JB)$

$P(JB)$

$= P(JBA) + P(JB\bar{A})$

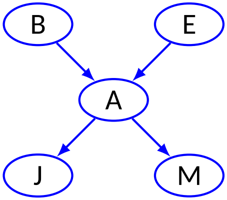
$= P(J|AB) \times P(AB) + P(J|\bar{A}B) \times P(\bar{A}B)$

$= P(J|A) \times P(AB) + P(J|\bar{A}) \times P(\bar{A}B)$

$= 0.9 \times 0.00095 + 0.05 \times 0.00005$

$= 0.00086$

$P(J|B)$



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T	0.70
F	0.01

Joint probability distribution: $P(JB)$

$P(JB)$

$$= P(JBA) + P(JB\bar{A})$$

$$= P(J|AB) \times P(AB) + P(J|\bar{A}B) \times P(\bar{A}B)$$

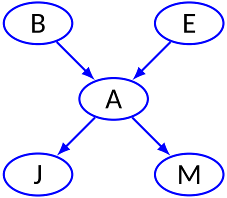
$$= P(J|A) \times P(AB) + P(J|\bar{A}) \times P(\bar{A}B)$$

$$= 0.9 \times 0.00095 + 0.05 \times 0.00005$$

$$= 0.00086$$

$P(J|B)$

$$= \frac{P(JB)}{P(B)} = \frac{0.00086}{0.001} = 0.86$$



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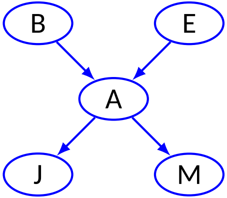
Inferences using belief networks

- **Diagnostic inferences (effects to causes)**
- **Given that JohnCalls, infer that** $P(Burglary|JohnCalls) = 0.016$

- **Causal inferences (causes to effects)**
- **Given Burglary, infer that**

  $P(JohnCalls|Burglary) = 0.86$

  $P(MaryCalls|Burglary) = 0.67$



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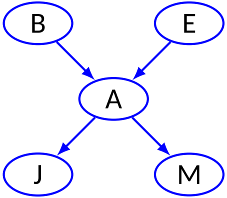
Inferences using belief networks

- **Inter-causal inferences (between causes to a common effect)**
- **Given Alarm, we have** $P(Burglary|Alarm) = 0.376$
- **Also, if it is given that Earthquake is true, then**
  $P(Burglary|Alarm \wedge Earthquake) = 0.003$

- **Mixed inferences**

  $P(Alarm|JohnCalls \wedge \neg Earthquake) = 0.003$



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Thank you!