

LOGICAL DEDUCTION IN AI

INFERRING BY RESOLUTION REFUTATION



Arijit Mondal & Partha P Chakrabarti
Indian Institute of Technology Kharagpur

Predicate Logic

Wherever Mary goes, so does the lamb. Mary goes to school. So the lamb goes to school.

No contractors are dependable. Some engineers are contractors. Therefore some engineers are not dependable.

All dancers are graceful. Ayesha is a student. Ayesha is a dancer. Therefore some student is graceful.

Every passenger is either in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

New Additions in Proposition (First Order Logic)

Function Symbols
Variables, Constants, Predicate Symbols and
New Connectors: $\exists$ (there exists), $\forall$ (for all)

Wherever Mary goes, so does the Lamb. Mary goes to School. So the Lamb goes to School.

Predicate: goes(x,y) to represent x goes to y

New Connectors: $\exists$ (there exists), $\forall$ (for all)

F1: $\forall x(\text{goes}(\text{Mary}, x) \rightarrow \text{goes}(\text{Lamb}, x))$

F2: $\text{goes}(\text{Mary}, \text{School})$ ✓ *ground instance*

G: $\text{goes}(\text{Lamb}, \text{School})$ ✓

To prove: $(F1 \wedge F2) \rightarrow G$ is always true

VALID

Inferencing in Predicate Logic

Domain: D

Constant Symbols: $M, N, O, P, \dots$

Variable Symbols: $x, y, z, \dots$

Function Symbols: $F(x), G(x, y), H(x, y, z)$

Predicate Symbols: $p(x), q(x, y), r(x, y, z)$

Connectors: $\sim, \wedge, \vee, \rightarrow, \exists, \forall$

Terms:

Well-formed Formula:

Free and Bound Variables:

Interpretation, Valid, Non-Valid, Satisfiable, Unsatisfiable

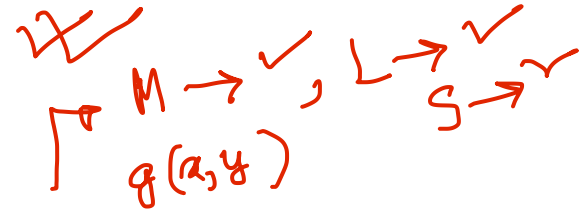
What is an Interpretation? Assign a domain set D , map constants, functions, predicates suitably. The formula will now have a truth value

Example:

$F1: \forall x(g(M, x) \rightarrow g(L, x))$

$F2: g(M, S)$

$G: g(L, S)$



Interpretation 1: $D = \{\text{Akash, Baby, Home, Play, Ratan, Swim}\}$, etc.,

Interpretation 2: $D = \text{Set of Integers, etc.,}$

How many interpretations can there be?

To prove Validity, means $(F1 \wedge F2) \rightarrow G$ is true under all interpretations

To prove Satisfiability means $(F1 \wedge F2) \rightarrow G$ is true under at least one interpretation

$\sim 2^n$ Truth Propositional Logic

INFINITE INTERPRETATIONS

Resolution Refutation for Propositional Logic

To prove validity of

$$F = ((F1 \wedge F2 \wedge \dots \wedge F_n) \rightarrow G)$$

we shall attempt to prove that

$$\sim F = (F1 \wedge F2 \wedge \dots \wedge F_n \wedge \sim G)$$

is unsatisfiable

→ FALSE, under all interpretations

Steps for Proof by Resolution

Refutation: $(a \vee b) \wedge (a \vee c) \wedge (\sim a)$

1. Convert of Clausal Form / Conjunctive Normal Form (CNF, Product of Sums).
2. Generate new clauses using the resolution rule.
3. At the end, either False will be derived if the formula $\sim F$ is unsatisfiable implying F is valid.

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP.

$$F1: (a \rightarrow (b \wedge c)) = (\sim a \vee b) \wedge (\sim a \vee c)$$

$$F2: \sim b, G: \sim a, \sim G: a \quad \neg a \vee (b \wedge c)$$

Clauses of Clause Form: $\sim F = (C1 \wedge C2 \wedge C3 \wedge C4)$

where: $C1: (\sim a \vee b)$

$C2: (\sim a \vee c)$

$C3: \sim b$

$C4: a$

To prove that $\sim F$ is False

Let $C1 = a \vee b$ and $C2 = \sim a \vee c$ $C3 = c' \vee c''$
then a new clause $C3 = b \vee c$ can be derived.

(Proof by showing that $((C1 \wedge C2) \rightarrow C3)$ is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C1 = a$ and $C2 = \sim a$ from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

RESOLUTION
RULE

Applying Resolution Refutation

Let $C1 = a \vee b$ and $C2 = \sim a \vee c$ → c
 then a new clause $C3 = b \vee c$ can be derived. $c \vee c$

(Proof by showing that $((C1 \wedge C2) \rightarrow C3)$ is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C1 = a$ and $C2 = \sim a$ from which **False** can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is **Sound** and **Complete**

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP. $e_1 \quad c_2$

$F1: (a \rightarrow (b \wedge c)) = (\sim a \vee b) \wedge (\sim a \vee c)$

$F2: \sim b$ ✓

$G: \sim a$

$\sim G: a$

$(F1 \wedge F2) \rightarrow G$
 $F1 \wedge F2 \wedge \neg G$

Clauses of Clause Form: $\sim F$
 $= (C1 \wedge C2 \wedge C3 \wedge C4)$

where: $C1: (\sim a) \vee b$

$C2: (\sim a \vee c)$ ✓

$C3: \sim b$ ✓

$C4: a$ ✓

To prove that $\sim F$ is False

New Clauses Derived

$C5: \sim a$ (Using $C1$ and $C3$)

$C6: \text{False}$ (using $C4$ and $C5$)

$C5: \sim a$ → **FALSE**

Example

Let $C1 = a \vee b$ and $C2 = \neg a \vee c$ then a new clause $C3 = b \vee c$ can be derived.

(Proof by showing that $((C1 \wedge C2) \rightarrow C3)$ is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C1 = a$ and $C2 = \neg a$ from which **False** can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is **Sound** and **Complete**

Rajesh either took the bus or came by cycle to class. If he came by cycle or walked to class he arrived late. Rajesh did not arrive late. Therefore he took the bus to class.

Handwritten notes illustrating the resolution process for the example:

Initial clauses:

- $F1: (bus \wedge \neg cycle) \vee (\neg bus \wedge cycle)$ (labeled "incl OR")
- $F2: (cycle \vee walk) \rightarrow late$
- $F3: \neg late$
- $G: bus$

Derived clauses and steps:

- $C6 \& C1: \bullet$
- $C1: cycle$
- $C1 \& C2: C8: late$
- $C8 \& C3: FALSE$
- $C2$
- $C1: (bus \vee cycle) \wedge (\neg bus \vee \neg cycle)$
- $F1: (bus \vee cycle) \wedge (\neg bus \vee \neg cycle)$
- $F2: \neg (cycle \vee walk) \vee late = (\neg cycle \wedge \neg walk) \vee late$
- $C5: (late \vee \neg cycle) \wedge (late \vee \neg walk)$
- $C4: (late \vee \neg walk)$
- $F3: \neg late$
- $C3: (late \vee \neg cycle) \wedge (late \vee \neg walk)$
- $\neg G: \neg bus$
- $C6: \neg bus$

Resolution Refutation for Predicate Logic

Given a formula F which we wish to check for validity, we first check if there are any free variables. We then quantify all free variables universally.

Create $F' = \sim F$ and check for unsatisfiability of F'

STEPS:

Conversion to Clausal (CNF) Form:

- Handling of Variables and Quantifiers, Ground Instances

Applying the Resolution Rule:

- Concept of Unification
- Principle of Most General Unifier (mgu)
- Repeated application of Resolution Rule using mgu

$F1: \forall x(\text{goes}(\text{Mary}, x) \rightarrow \text{goes}(\text{Lamb}, x))$ *constants*

$F2: \text{goes}(\text{Mary}, \text{School})$ *variables*

$G: \text{goes}(\text{Lamb}, \text{School})$ *predicates*

To prove: $(F1 \wedge F2) \rightarrow G$ is valid *functions*

CONVERSION TO CLAUSAL FORM IN PREDICATE LOGIC

- Remove implications and other Boolean symbols converting to equivalent forms using $\sim, \vee, \wedge$
- Move negatives ($\sim$) inwards as close as possible
- Standardize (Rename) variables to make them unambiguous
- Remove Existential Quantifiers by an appropriate new function/constant symbol taking into account the variables dependent on the quantifier
(Skolemization)
- Drop Universal Quantifiers
- Distribute $\vee$ over $\wedge$ and convert to CNF

$F1 \wedge F2 \wedge \neg G \quad \forall x \{ \neg \text{goes}(\text{Mary}, x) \vee \text{goes}(\text{Lamb}, x) \}$
 $C1: \neg \text{goes}(\text{Mary}, x) \vee \text{goes}(\text{Lamb}, x)$
 $C2: \text{goes}(\text{Mary}, \text{School})$ $C3: \neg \text{goes}(\text{Lamb}, \text{School})$
 $C4: \text{goes}(\text{Lamb}, \text{School}) \rightarrow \text{FALSE}$

Conversion to Clausal Form

1. Remove implications and other Boolean symbols converting to equivalent forms using $\sim$, $\vee$, $\wedge$
2. Move negates ($\sim$) inwards as close as possible ✓
3. Standardize (Rename) variables to make them unambiguous
4. Remove Existential Quantifiers by an appropriate new function / constant symbol taking into account the variables dependent on the quantifier
(Skolemization) ✓
5. Drop Universal Quantifiers ✓
6. Distribute $\vee$ over $\wedge$ and convert to CNF ✓

$$\forall x (\forall y (\text{student}(y) \rightarrow \text{likes}(x, y)) \rightarrow (\exists z (\text{likes}(z, x))))$$

$$\forall x \{ \forall y (\text{student}(y) \rightarrow \text{likes}(x, y)) \rightarrow \exists z (\text{likes}(z, x)) \}$$

$$\forall x \{ \neg (\forall y (\text{student}(y) \rightarrow \text{likes}(x, y))) \vee \exists z (\text{likes}(z, x)) \}$$

$$\forall x \{ \neg (\forall y (\neg \text{student}(y) \vee \text{likes}(x, y))) \vee \exists z (\text{likes}(z, x)) \}$$

$$\forall x \{ \neg (\neg \text{student}(y) \vee \neg \text{likes}(x, y)) \vee \exists z (\text{likes}(z, x)) \}$$

$$F(x), G(x)$$

$$\forall x \{ (\text{student}(F(x)) \wedge \neg \text{likes}(x, F(x))) \vee \text{likes}(G(x), x) \}$$

Substitution, Unification, Resolution

Consider clauses:

- C1: $\sim \text{studies}(x, y) \vee \text{passes}(x, y)$ ✓
- C2: $\text{studies}(\text{Madan}, z)$
- C3: $\sim \text{passes}(\text{Chetan}, \text{Physics})$ ✓
- C4: $\sim \text{passes}(w, \text{Mechanics})$ ✓

What new clauses can we derive by the resolution principle?

Ground Clause and a more general clause

Concept of substitution / unification and the Most General Unifier (mgu)

Resolution Rule for Predicate Calculus: Repeated Application of Resolution using mgu

$$\exists x \forall y p(x, y) \Rightarrow \neg \forall y p(A, y) \Rightarrow p(A, y)$$

$$\forall x \exists y p(x, y) \Rightarrow \forall x p(x, f(x))$$

$$\Rightarrow p(x, f(x))$$

SKOLEM FN

$$\begin{array}{l} C1 \& C2: C' \text{ passes}(\text{Madan}, z) \\ C1 \& C2: C'' \text{ passes}(\text{Madan}, \text{Physics}) \\ \hline C' \text{ is more general than } C'' \end{array}$$

$$\begin{array}{l} C1 \& C4: y/\text{Mechanics} \\ \quad \neg \text{studies}(x, \text{Mechanics}) \end{array} \Bigg] C'''$$

$$C2 \& C''': \text{FALSE}$$

most general unifier

Examples

F1: $\forall x(\text{contractor}(x) \rightarrow \sim \text{dependable}(x))$ $c(x)$

F2: $\exists x(\text{engineer}(x) \wedge \text{contractor}(x))$ $d(x)$

G: $\exists x(\text{engineer}(x) \wedge \sim \text{dependable}(x))$ $e(x)$

F1: $\forall x \{ \neg c(x) \vee \neg d(x) \}$ $c1$

F2: $e(A) \wedge c(A)$

$\neg G$: $\neg \exists x (e(x) \wedge \neg d(x))$
 $\forall x (\neg e(x) \vee d(x))$

$c2$ $c3$
 $d(A)$ $c5$

$\neg c(A)$ $c6$

FALSE

F1: $\forall x(\text{dancer}(x) \rightarrow \text{graceful}(x))$

F2: student(Ayesha), F3: dancer(Ayesha)

G: $\exists x(\text{student}(x) \wedge \text{graceful}(x))$

F1: $\forall x \{ \neg d(x) \vee g(x) \}$ $c1$
 $d(A)$ $c3$

F2: $s(A)$ $c2$

F3: $d(A)$ $c3$
 $\neg G$: $\neg \exists x \{ s(x) \wedge g(x) \}$
 $\forall x \{ \neg s(x) \vee \neg g(x) \}$ $c4$

0. F1, F2, ...
 $\wedge \neg G$

1. Convert to
 clausal Form

2. Resolution Rule
UNSATISFIABILITY

x/A
 $\neg g(A)$
 x/A
 $\neg d(A)$

FALSE

Thank you