

# LOGICAL DEDUCTION IN AI

## PREDICATE LOGIC FUNDAMENTALS



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# Predicate Logic

Wherever Mary goes, so does the lamb. Mary goes to school. So the lamb goes to school.

No contractors are dependable. Some engineers are contractors. Therefore some engineers are not dependable.

All dancers are graceful. Ayesha is a student. Ayesha is a dancer. Therefore some student is graceful.

Every passenger is either in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

New Additions in Proposition (First Order Logic)

Variables, Constants, Predicate Symbols and New Connectors:  $\exists$  (there exists),  $\forall$  (for all)

Function

all some

Wherever Mary goes, so does the Lamb. Mary goes to School. So the Lamb goes to School.

Predicate: goes(x,y) to represent x goes to y

New Connectors:  $\exists$  (there exists),  $\forall$  (for all)

F1:  $\forall x(\text{goes}(\text{Mary}, x) \rightarrow \text{goes}(\text{Lamb}, x))$  ← x is bound by  $\forall x$  quantified

F2: goes(Mary, School) ✓

G: goes(Lamb, School) ✓

To prove:  $(F1 \wedge F2) \rightarrow G$  is always true

# Use of Quantifiers

## EXAMPLES:

Someone likes everyone ✓

Everyone likes someone ✓

There is someone whom everyone likes ✓

Everyone likes everyone

If everyone likes everyone then someone likes everyone

If there is a person whom everyone likes then that person likes himself

## LAWS of NEGATION:

Exercise

$$\neg \exists x (p(x)) \equiv \forall x (\neg p(x))$$

$$\neg \forall x (p(x)) \equiv \exists x (\neg p(x))$$

$\exists$   $\neg$   $\forall$   $\neg$   $\exists$   
 $i=1$   $i$   $i$   $i$   $i$

$likes(x, y) : x \text{ likes } y$

$$\exists x (\forall y (likes(x, y))) \quad F1$$

$$\forall x (\exists y (likes(x, y))) \quad \checkmark$$

$y$  depends on  $x$

$$\exists y \forall x (likes(x, y)) \quad \checkmark$$

$y$  is independent of  $x$

$$\forall x \forall y (likes(x, y)) \quad \forall y \forall x likes(x, y)$$

$$F2 \rightarrow F1 \quad \checkmark$$

scope rules ✓

$$\exists x_1 (\forall y_1 (likes(x_1, y_1))) \rightarrow (\forall x_2 \forall y_2 (likes(x_2, y_2)))$$

De Morgan's laws

$F2$

# Use of Function Symbols

If  $x$  is greater than  $y$  and  $y$  is greater than  $z$  then  $x$  is greater than  $z$ .

The age of a person is greater than the age of his child.

Therefore the age of a person is greater than the age of his grandchild.

The sum of ages of two children are never more than the sum of ages of their parents.

variables  
Constant  $\rightarrow$  Functions  
propositions  $\rightarrow$  predicates  
 $\forall, \exists$

$g(x, y)$   $x$  is greater than  $y$   
 $\forall x \forall y \forall z ((g(x, y) \wedge g(y, z)) \rightarrow g(x, z))$

Constant Symbol  $A \rightarrow B(x)$   
 $\rightarrow$  Age( $x$ )  $\rightarrow$  Function Symbol  $\rightarrow$  variable  
 $\rightarrow$  child( $x, y$ ):  $x$  is a child of  $y$

$\forall x \forall y (\text{child}(x, y) \rightarrow g(\text{Age}(y), \text{Age}(x)))$

Age( $x$ )  $\rightarrow$  returns a value  
child( $x, y$ )  $\rightarrow$  returns TRUE or FALSE

Sum( $x, y$ )  $\rightarrow$  Function Symbol  
parent( $x, y$ ): use the child predicate

# Variables and Predicate / Function Symbols

Variables, Free variables, Bound variables

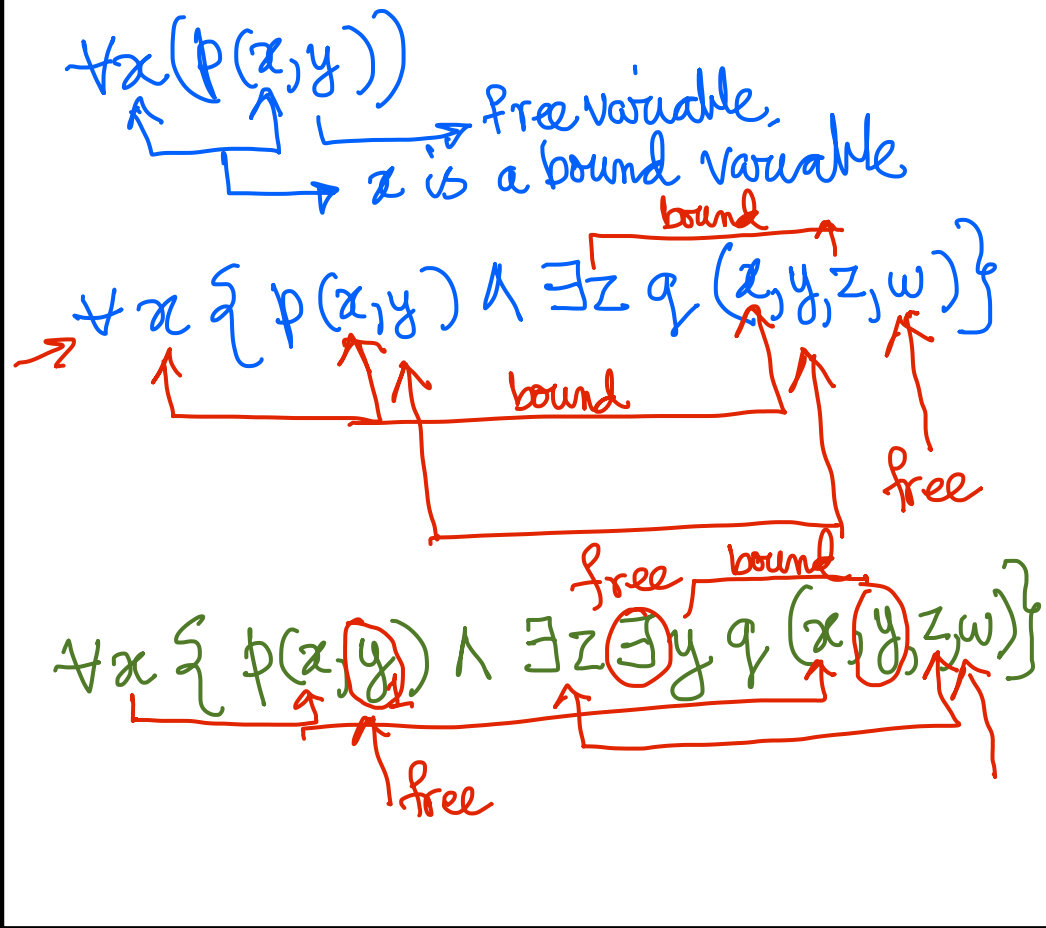
Symbols – proposition symbols, constant symbols, function symbols, predicate symbols

Variables can be quantified in first order predicate logic

Symbols cannot be quantified in first order predicate logic

Interpretations are mappings of symbols to relevant aspects of a domain

~~$\exists x \{ \forall x \{ p(x) \} \}$~~   
Not in predicate logic



# Terminology for Predicate Logic

Domain:  $D$

Constant Symbols:  $M, N, O, P, \dots$

Variable Symbols:  $x, y, z, \dots$  ✓

Function Symbols:  $F(x), G(x, y), H(x, y, z)$

Predicate Symbols:  $p(x), q(x, y), r(x, y, z)$

Connectors:  $\sim, \wedge, \vee, \rightarrow, \exists, \forall$

Terms: ✓

Well-formed Formula: ✓

Free and Bound Variables: ✓

Interpretation, Valid, Non-Valid,  
Satisfiable, Unsatisfiable

SYNTAX

SEMANTICS

$p(x)$  where  $x$  is free  
in  $p$   
 $\forall x p(x)$   $\exists x p(x)$

$D$ : Domain  $D$  will be specified for every interpretation

Term: - variables are terms  
- constant symbols are terms

if  $t_1, t_2, \dots, t_k$  is a term

and  $F(x_1, x_2, \dots, x_k)$  is a  $F$  in symbol  
then  $F(t_1, t_2, \dots, t_k)$  is also a term

if  $p(x_1, x_2, \dots, x_k)$  is a predicate symbol

$p(t_1, t_2, \dots, t_k)$

WFF: propositions are wffs,

$\textcircled{p}$  is a WFF  
 $p$  &  $q$  are WFF  
 $\neg p$  is a WFF  
 $p \wedge q, p \vee q, p \rightarrow q$

# Validity, Satisfiability, Structure

F1:  $\forall x(\text{goes}(\text{Mary}, x) \rightarrow \text{goes}(\text{Lamb}, x))$

F2:  $\text{goes}(\text{Mary}, \text{School})$

G:  $\text{goes}(\text{Lamb}, \text{School})$

To prove:  $(F1 \wedge F2) \rightarrow G$  is always true

Is the same as:

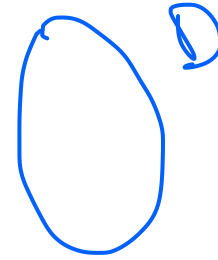
F1:  $\forall x(\text{www}(\text{M}, x) \rightarrow \text{www}(\text{L}, x))$

F2:  $\text{www}(\text{M}, \text{S})$

G:  $\text{www}(\text{L}, \text{S})$

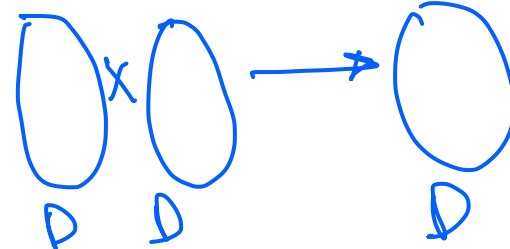
To prove:  $(F1 \wedge F2) \rightarrow G$  is always true

predicate symbol



Interpretation  
MAPPING

$F: D \times D \rightarrow D$       $F(x, y)$



$F: D^n \rightarrow D$

$p: D^n \rightarrow \{T, F\}$

$p(x, y) \in D$

$\{T, F\}$  ✓  
← relation

# Interpretations

**What is an Interpretation?** Assign a domain set  $D$ , map constants, functions, predicates suitably. The formula will now have a truth value

Example:

$F1: \forall x(g(M, x) \rightarrow g(L, x))$

$F2: g(M, S)$

$G: g(L, S)$

Interpretation 1:  $D = \{\text{Akash, Baby, Home, Play, Ratan, Swim}\}$ , etc.,

Interpretation 2:  $D = \text{Set of Integers, etc.}$

How many interpretations can there be?

To prove Validity, means  $(F1 \wedge F2) \rightarrow G$  is true under all interpretations

To prove Satisfiability means  $(F1 \wedge F2) \rightarrow G$  is true under at least one interpretation

→ if it is true under SOME interpretation

$D$ : Assign the Domain  
 $F$ : Assign constants or  $F^n$ 's from the Domain  
 $P$ : Assign a specific relation from the Domain

$M$ : Home,  $L$ : Akash,  $S$ : Akash ✓  
 $g(x, y)$ : for all pairs in  $D$  we have say whether  $g(x, y)$  is T/F  
↑

$D$ : set of Integers ✓  $1, M, S$  ✓  
 $g(x, y)$ :  $x$  divides  $y$  ✓  
 $F1 \wedge F2 \rightarrow G$

A formula is said to be valid if it is true FOR ALL interpretations



# In Its Power Lies Its Limitations

Russell's Paradox (The barber shaves all those who do not shave themselves. Does the barber shave himself?)

- There is a single barber in town.
- Those and only those who do not shave themselves are shaved by the barber.
- Who shaves the barber?

Checking Validity of First order logic is undecidable but partially decidable (semi-decidable) {Robinson's Method of Resolution Refutation}

Higher order predicate logic - can quantify symbols in addition to quantifying variables.

$$\forall p((p(0) \wedge (\forall x(p(x) \rightarrow p(S(x)))) \rightarrow \forall y(p(y)))$$

NOT IN 1st order logic

Power? Computation

Predicate Logic can model any computable function TURING M/C

→ Undecidability | HALTING

Barber shaves himself X  
Someone else shaves the Barber X  
Unsolvable Problem ✓

if the answer is YES then there is a method SEMI-DECIDABLE ✓

HIGHER ORDER LOGIC ↑ Which Theorem  
→ what this query tries to say

Thank you