# Propositional logic: Knight, Knave, Spy

$q$ = B ≠ spy  $r$ = C = Knave  $p$ = A is Knight

- In an island there are three types of people - knave, knight, spy. A knight always tells the truth. A knave always lies. A spy can either tell the truth or lie. A visitor in the island meets three people A, B, and C. A says, "I am knight." B says, "I am not a spy." C says, "I am a knave." It is known that A, B, and C are of different types of people. Define the set of atomic propositions, model the problem using propositional logic, then determine the type of A, B and C.

q

| | A | B | C | $p$ | $q$ | $r$ |
|---|---|---|---|---|---|---|
| ✗ → | Knight | Knave | Spy | ✓ | ✗ | ✓ |
| ✗ → | Knight, | Spy | Knave | ✓ | ✓ | ✗ |
| ✓ → | Knave | Knight | Spy | ✓ | ✓ | ✓ |

$(A = \text{Knight})\ p \ \vee$

$(A = \text{Knave})\ \neg p$

A - Knight - $x_1$

A Knave - $x_2$

# Propositional Logic: Glasses

$\neg p \vee q \equiv p \to q \equiv \neg q \to \neg p$

- You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:
  - If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.
  - If my glasses are on the kitchen table, then I saw them at breakfast.
  - I did not see my glasses at breakfast.
  - I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
  - If I was reading the newspaper in the living room then my glasses are on the coffee table.
- Where are the glasses?

$n \to k$

$\boxed{p \to q}, \quad q \to r, \quad \neg r, \quad s \vee p, \quad s \to t$

$\neg p$

$q \to r \equiv \neg q \vee r$

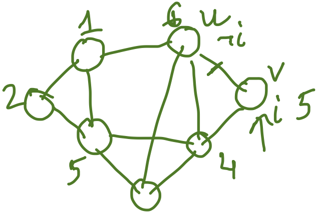
$\neg r \to \neg q$

$s$

$t$

# SAT modeling: Graph coloring

- Given a graph $G = (V, E)$ and $n$ different colors, is it possible to color the vertices of the graph using the given set of colors such that two vertices that share an edge have different colors? Find the chromatic number - minimum number of colors needed to color the vertices.




$(1,2,3) \leftarrow x$

$v_1, v_2, \dots v_n$

$\rightarrow e(v,u) \Rightarrow \boxed{\begin{array}{c} v_i \rightarrow \neg u_i \\ \hline u_i \rightarrow \neg v_i \end{array}}$

$e(u,v) \Rightarrow$

$\rightarrow \boxed{v_1 \vee v_2 \vee \dots \vee v_n} \leftarrow$

$n \xleftrightarrow{\boxed{BS}} 1$

Assign at most one color to a node

# SAT modeling: Multiplication

$(21)_2 = \underline{10101}$

$7, 3$

$111 \text{ — } 3 \quad 11$

- **Develop a SAT model for the following problem. Given an** $(m+n)$**-bit binary integer** $z = (z_{m+n} \ldots z_2 z_1)_2$**, do there exist integers** $x = (x_m \ldots x_1)_2$ **and** $y = (y_n \ldots y_1)_2$ **such that** $z = x \times y$**? Write the SAT constraints for** $m = 2, n = 3$**.**

# SAT modeling: Multiplication

- **Develop a SAT model for the following problem. Given an $(m+n)$-bit binary integer $z = (z_{m+n} \ldots z_2 z_1)_2$, do there exist integers $x = (x_m \ldots x_1)_2$ and $y = (y_n \ldots y_1)_2$ such that $z = x \times y$? Write the SAT constraints for $m = 2, n = 3$.**

$$
\begin{array}{ccccc}
 & & y_3 & y_2 & y_1 \\
 & \times & & x_2 & x_1 \\
\hline
 & & a_3 & a_2 & a_1 \\
 & b_3 & b_2 & b_1 & \\
\hline
c_3 & c_2 & c_1 & \text{carry} & \\
\hline
z_5 & z_4 & z_3 & z_2 & z_1
\end{array}
$$

# SAT modeling: Multiplication

- **Develop a SAT model for the following problem. Given an $(m+n)$-bit binary integer $z = (z_{m+n} \ldots z_2 z_1)_2$, do there exist integers $x = (x_m \ldots x_1)_2$ and $y = (y_n \ldots y_1)_2$ such that $z = x \times y$? Write the SAT constraints for $m = 2, n = 3$.**

$$
\begin{array}{ccccc}
 & & y_3 & y_2 & y_1 \\
 & \times & & x_2 & x_1 \\
\hline
 & & a_3 & a_2 & a_1 \\
 & b_3 & b_2 & b_1 & \\
\hline
c_3 & c_2 & c_1 & & \\
\hline
z_5 & z_4 & z_3 & z_2 & z_1
\end{array}
$$

$$(a_3 a_2 a_1)_2 = (y_3 y_2 y_1)_2 \times x_1$$
$$(b_3 b_2 b_1)_2 = (y_3 y_2 y_1)_2 \times x_2$$

# SAT modeling: Multiplication

$(a \vee b \vee c) \wedge (\bar{a} \vee b \vee c)$ ✓

- **Develop a SAT model for the following problem. Given an $(m+n)$-bit binary integer $z = (z_{m+n} \ldots z_2 z_1)_2$, do there exist integers $x = (x_m \ldots x_1)_2$ and $y = (y_n \ldots y_1)_2$ such that $z = x \times y$? Write the SAT constraints for $m = 2, n = 3$.**

$$
\begin{array}{ccccc}
 & & y_3 & y_2 & \boxed{y_1} \\
 & \times & & x_2 & \boxed{x_1} \\
\hline
 & & a_3 & a_2 & a_1 \\
 & b_3 & b_2 & b_1 & \\
\hline
c_3 & c_2 & c_1 & & \\
\hline
z_5 & z_4 & z_3 & z_2 & z_1
\end{array}
$$

$$(a_3 a_2 a_1)_2 = (y_3 y_2 y_1)_2 \times x_1$$ ✓

$$(b_3 b_2 b_1)_2 = (y_3 y_2 y_1)_2 \times x_2$$ ✓

$$
\begin{aligned}
z_1 &= a_1 \\
(c_1 z_2)_2 &= a_2 + b_1 \\
(c_2 z_3)_2 &= a_3 + b_2 + c_1 \\
(c_3 z_4)_2 &= b_3 + c_2 \\
z_5 &= c_3
\end{aligned}
$$

Adder

$(x_1 \wedge y_1) \leftrightarrow z_1$ ✓

$x_1 \wedge y_2 \leftrightarrow a_2$

$x_1 \wedge y_3 \leftrightarrow a_3$

$z_2 \leftrightarrow a_2 \oplus b_1$

$c_1 \leftrightarrow (a_2 \wedge b_1)$

$p \quad q$

$(b + c)(\bar{b} + \bar{c})$

$a \rightarrow b \oplus c \equiv \neg a \vee (b\bar{c} + \bar{b}c) \equiv \boxed{\bar{a} + b\bar{c} + \bar{b}c}$

$(\neg a + pq) \equiv (\neg a + p)(\neg a + q)$

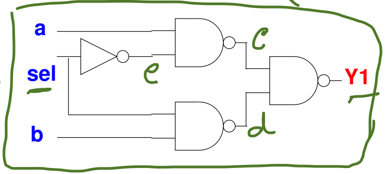
# SAT modeling: Circuit equivalence

- **Are these two circuits equivalent?**

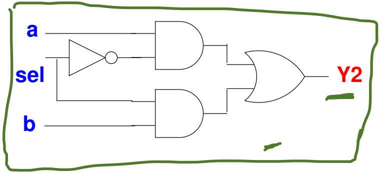
MUX

$\neg\boxed{y_1 \leftrightarrow y_2}$

CNF




CNF




$sel \leftrightarrow \neg e$

$\neg(a \wedge e) \leftrightarrow c$

$\neg(b \wedge sel) \leftrightarrow d$

$\neg(c \wedge d) \leftrightarrow y_1$

$X_{CNF}$

y2

$\neg(y_1 \leftrightarrow y_2),\ Y_{CNF},\ X_{CNF}$

UNSAT