Lect 25 : 10th September

Laplace Transform

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

For any CT signal $x(t)$, it's **Bilateral** Laplace-transform is the function:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}.$$

The Region of Convergence of $X(s)$ is defined to be

$$\mathtt{ROC} = \{\mathtt{s} \mid \int_{-\infty}^{\infty} |\mathtt{x(t)}||\mathtt{e}^{-\mathtt{st}}|\mathtt{dt} = \int_{-\infty}^{\infty} |\mathtt{x(t)}|\mathtt{e}^{-\mathtt{Re(s)t}}\mathtt{dt} < \infty\}.$$

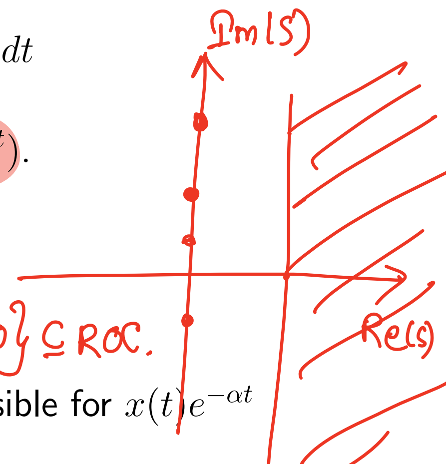
$s_1 = \alpha_1 + j\omega_1$
$s_2 = \alpha_1 + j\omega_2$
if $s_1 \in$ ROC,
then $s_2 \in$ ROC

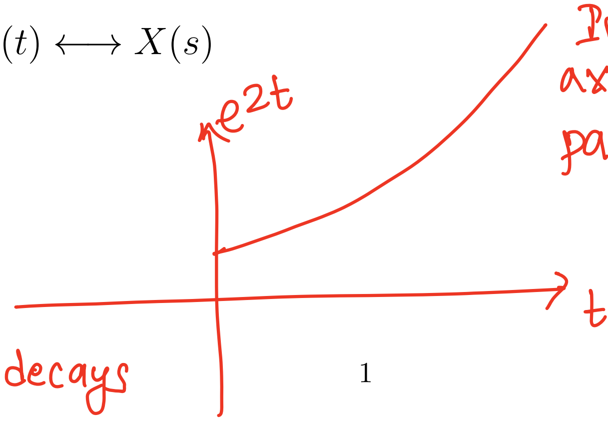
$s = \alpha + j\omega$, $|e^{-st}| = |e^{-\alpha t}|$

- Letting $\alpha = \mathtt{Re}(s)$ and $\omega = \mathtt{Im}(s)$, we have:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-\alpha t}e^{-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} (x(t)e^{-\alpha t})e^{-j\omega t}dt = \mathtt{FT}(x(t)e^{-\alpha t}).$$

- Therefore, if $s = \alpha + \omega j \in \mathtt{ROC}$, $X(s) = \mathtt{FT}(x(t)e^{-\alpha t})$.
- If $\alpha = 0$, $X(s) = \mathtt{FT}(x(t))$. assuming that $\{s \mid Re(s) = 0\} \subseteq$ ROC.
- Even when $x(t)$ is not absolutely integrable, it may be possible for $x(t)e^{-\alpha t}$ to be absolutely integrable if α is sufficiently large.
- Notation: $x(t) \longleftrightarrow X(s)$





Imaginary axis not part of ROC.

$a = 2$

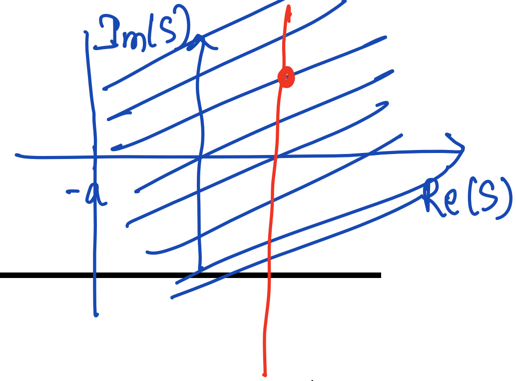
ROC $= \{s \mid Re(s) > 2\}$

$x(t) = e^{2t}u(t)$

$X(s) = \dfrac{1}{s-2}$

Fourier transform does not exist.

[$e^{2t}u(t)e^{-st}$ decays when $Re(s) > 2$



Laplace Transform

- Example 1: What is the Laplace Transform and ROC of $x(t) = e^{-at}u(t)$ where $a \in \mathbb{R}$?

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st}\, dt = \int_{0}^{\infty} e^{-(a+s)t}\, dt = \frac{-1}{(a+s)} e^{-(a+s)t} \Big|_0^{\infty}$$

$$= \frac{-1}{(a+s)} (0 - 1) \quad \text{when} \quad Re(a+s) > 0$$

$$= \frac{1}{a+s}, \quad \text{when} \quad Re(s) > -a$$

$$ROC = \{ s \mid Re(s) > -a \}$$

- Example 2: What is the Laplace Transform and ROC for $x(t) = -e^{-at}u(-t)$?

$$X(s) = \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st}\, dt = \int_{-\infty}^{0} -e^{-(a+s)t}\, dt$$

$$= \frac{1}{a+s} e^{-(a+s)t} \Big|_{-\infty}^{0} = \frac{1}{a+s} \quad \text{when} \quad Re(a+s) < 0$$

$$Re(s) < -a$$

- Moral: Laplace-Transform without ROC is meaningless!
- Moral: ROC consists of vertical lines.

Poles and Zeros

- We almost exclusively work with rational Laplace transforms:

$$X(s) = \frac{a_0(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)},$$

  where $z_1, \ldots, z_m \in \mathbb{C}$ are the zeros and $p_1, \ldots, p_n \in \mathbb{C}$ are the poles.

- Location of the poles are important for ROC and stability

- Example: What are the poles and zeros of $X(s) = \frac{2s-1}{s^2+2s-1}$?

zeros: $s = 0.5$

poles: $s = -1+\sqrt{2}$ & $s = -1-\sqrt{2}$ are the two poles of $X(s)$.

- Determine the Laplace Transform of $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$ with its ROC.

$= e^{-2t}u(t) + e^{-t}\frac{1}{2}\left[e^{j3t} + e^{-j3t}\right]u(t)$

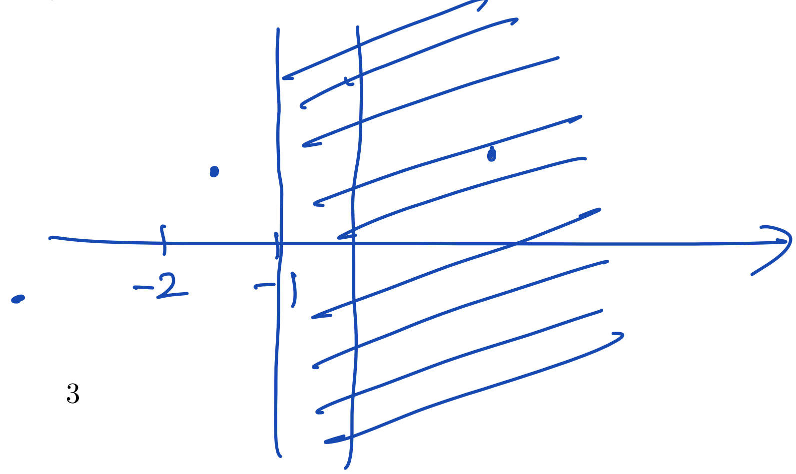
$$x(t) = \underbrace{e^{-2t}u(t)}_{\downarrow} + \underbrace{\frac{1}{2}e^{-(1+j3)t}u(t)}_{\downarrow} + \underbrace{\frac{1}{2}e^{-(1-j3)t}u(t)}$$

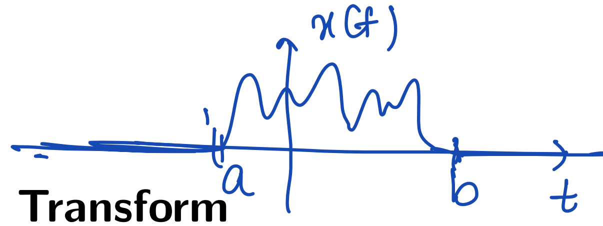
$$= \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1+j3} + \frac{1}{2}\frac{1}{s+1-j3}$$

ROC: $Re(s) > -2$, $Re(s) > -1$, $Re(s) > -1$

ROC of $x(t)$ is

$Re(s) > -1$





Properties of ROC for Laplace Transform

$\left[x(t) = 0 \text{ for } t < a \text{ and } t > b \right.$

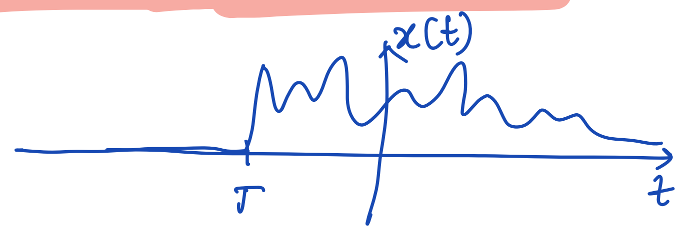
- Property 1: ROC does not contain any poles.
- Property 2: If $x(t)$ is of finite duration and $x(t)$ is absolutely integrable, i.e., $\int_a^b |x(t)| dt < \infty$, then ROC is the entire $\mathbb{C}$.
  - To show this, let $s \in \mathbb{C}$ with $\mathtt{Re}(s) = \alpha$. Then:

$$\int_{-\infty}^{\infty} |x(t)| e^{-\alpha t} dt = \int_a^b |x(t)| e^{-\alpha t} dt$$
$$\leq \max(e^{-\alpha t} : t \in [a, b]) \int_a^b |x(t)| dt < \infty.$$

- Property 3: If $x(t)$ is right-sided and $s_0 \in$ ROC, then the whole half-plane $\{s : \mathtt{Re}(s) > \mathtt{Re}(s_0)\}$ is in ROC.

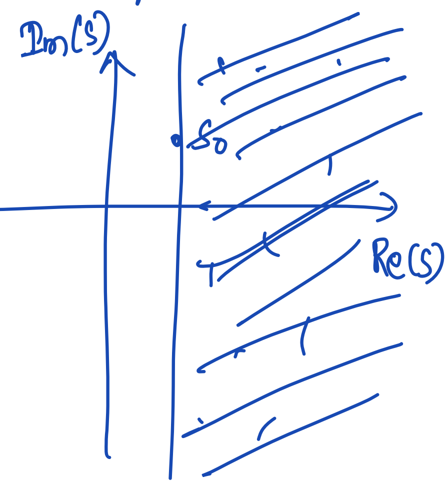
$\exists\, T$ s.t. $x(t) = 0$ for $t < T$

Let $\bar{s} \in \mathbb{C}$ s.t. $\mathrm{Re}(\bar{s}) > \mathrm{Re}(s_0)$



$$\int_{-\infty}^{\infty} |x(t)| e^{-\mathrm{Re}(\bar{s})} dt$$

$$= \int_{T}^{\infty} |x(t)| e^{-\mathrm{Re}(\bar{s})} dt$$



$$= \int_{T}^{\infty} |x(t)| e^{-\mathrm{Re}(s_0)t} \left[e^{\underbrace{(\mathrm{Re}(s_0) - \mathrm{Re}(\bar{s}))}_{-ve.}t} \right] dt$$

$$\leq \underbrace{e^{(\mathrm{Re}(s_0) - \mathrm{Re}(\bar{s}))T}}_{\text{finite}} \underbrace{\int_{T}^{\infty} |x(t)| e^{-\mathrm{Re}(s_0)t} dt}_{< \infty \text{ since } s_0 \in \mathrm{ROC}.}$$

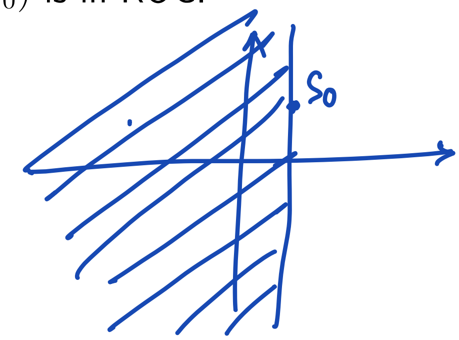
$$< \infty.$$

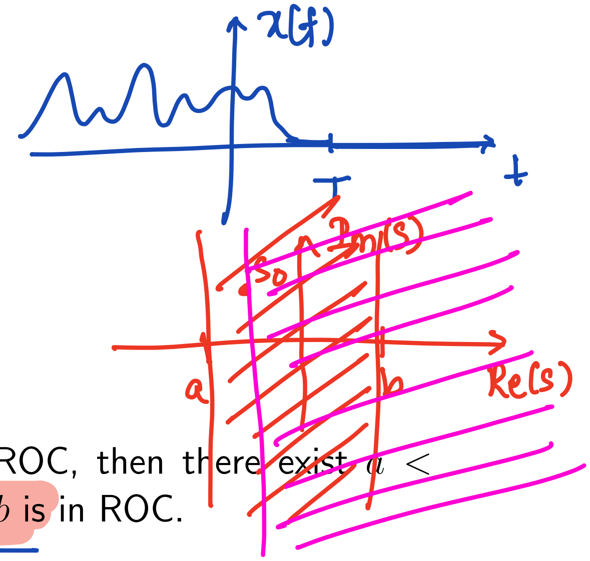
LECTURE 26: 11th September

Properties of ROC for Laplace Transform

$\exists\, T$ s.t $x(t) = 0 \;\; \forall\, t > T$

- Property 4: If $x(t)$ is left-sided and $s_0 \in$ ROC, then the half-plane s : $\mathrm{Re}(s) < \mathrm{Re}(s_0)$ is in ROC.



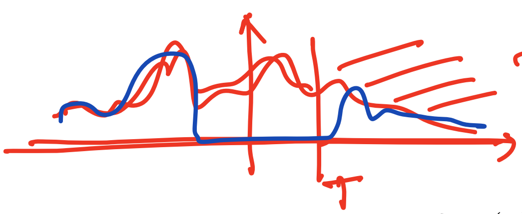


- Property 5: If $x(t)$ is two-sided and s_0 is in ROC, then there exist $a < \mathrm{Re}(s_0) < b$ such that the strip $s : a < \mathrm{Re}(s) < b$ is in ROC.
- Property 6: If $X(s)$ is rational, then ROC is either bounded between two poles or extends to infinity.
- Property 7: If $X(s)$ is rational and x(t) is right-sided, then the ROC is right-half plane to the right-most pole.

we can express

$X(S) = X^+(S) + X^-(S)$

$x(t) = x^+(t) + x^-(t)$, where $x^+(t) = \begin{cases} 0 & \forall t < T \\ x(t) & \forall t \geq T \end{cases}$



$x^-(t) = \begin{cases} 0, & \forall t \geq T \\ \underline{x(t)}, & \forall t < T \end{cases}$

- Property 8: If $X(s)$ is rational and x(t) is left-sided, then the ROC is left-half plane to the left-most pole.

For a right-sided signal $x^+(t)$, ROC $= \{s \mid \mathrm{Re}(s) > \sigma_+\}$

$x^-(t)$, ROC $= \{s \mid \mathrm{Re}(s) < \sigma_-\}$

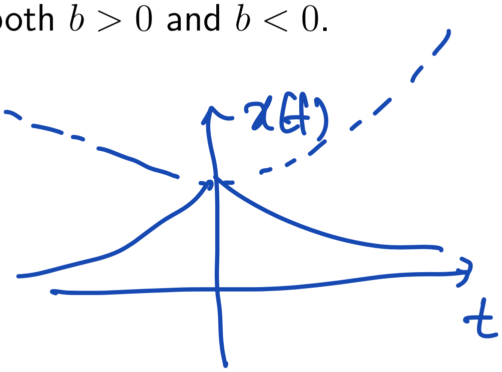
Case 1: $\sigma_+ > \sigma_-$: ROC of $x(t) = \emptyset$

Case 2: $\sigma_+ < \sigma_-$: ROC of $x(t) = \{s \mid \mathrm{Re}(s) \in (\sigma_+, \sigma_-)\}$

Laplace Transform

Determine the Laplace transform of $x(t) = e^{-b|t|}$ for both $b > 0$ and $b < 0$.

Case 1 : $b > 0$



$$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$$

$$X(s) = \frac{1}{s+b} - \frac{1}{s-b}$$

$Re(s) > -b$ ROC: $Re(s) < b$.

$$= \frac{1}{s+b} - \frac{1}{s-b}$$

$$-e^{bt}u(-t) \longleftrightarrow \frac{1}{s-b},$$

$$= \frac{-2b}{s^2 - b^2}, \quad \text{ROC: } \{s \mid -b < Re(s) < b\}.$$

ROC: $Re(s) < b$.

Case 2 : $b < 0$

$$e^{-bt}u(t) \longleftrightarrow \frac{1}{s+b}, \quad \text{ROC: } \{s \mid Re(s) > -b\}$$

+ve

$$e^{bt}u(t) \longleftrightarrow \frac{-1}{s-b}, \quad \text{ROC: } \{s \mid Re(s) < b\}$$

-ve

$X(s)$ does not exist

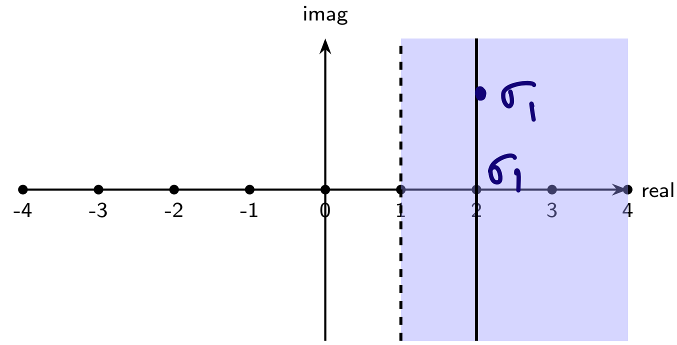
since $\{s \mid Re(s) > -b\}$

$\cap \{s \mid Re(s) < b\}$

is an empty set.

Inverse Laplace Transform

- Suppose that the line $\{s|\mathtt{Re}(s) = \sigma_1\} \in ROC$



- We know that along this line $X(\sigma + j\omega) = \mathtt{FT}(e^{-\sigma t}x(t))$.
- Therefore,

$$x(t) = e^{\sigma t}\mathtt{IFT}(X(\sigma + j\omega)) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X((\sigma + j\omega))e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{(\sigma+j\omega)t}\, d\omega$$

let $s = \sigma + j\omega$, $ds = j\, d\omega \Rightarrow d\omega = \frac{ds}{j}$

$=$

Inverse Laplace Transform

- For rational transfer functions $X(s) = \frac{q(s)}{p(s)}$, we proceed as in the case of inverse Fourier transform:

  a. We perform partial fraction on

$$X(s) = \frac{q(s)}{p(s)} = \frac{A}{s+p_1} + \frac{B}{s+p_2} + \cdots$$

  b. If pole p_i is to the left side of ROC, we use $e^{-p_i t}u(t) \longleftrightarrow \frac{1}{s+p_i}$

  c. If pole p_i is to the right side of ROC, we use $-e^{-p_i t}u(-t) \longleftrightarrow \frac{1}{s+p_i}$

- Example: What is the Laplace Transform inverse of $X(s) = \frac{s}{s^2+5s+6}$ given that ROC$= \{s \mid -3 < \mathrm{Re}(s) < -2\}$?

$(s+2)(s+3)$

$$X(s) = \frac{A}{s+2} + \frac{B}{s+3} = \frac{(A+B)s + (2B+3A)}{(s+2)(s+3)},$$

$$\begin{cases} A+B = 1 \\ 3A+2B = 0 \end{cases}$$

$3A + 2(1-A) = 0$

$\Rightarrow A + 2 = 0$

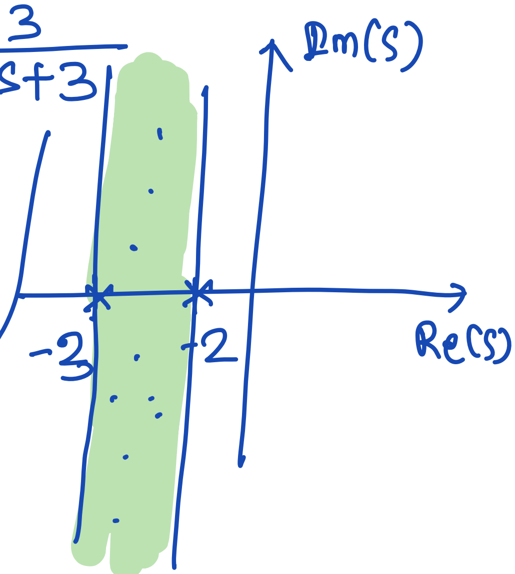
$\Rightarrow A = -2$

$\Rightarrow B = 3$

$$= \underbrace{\frac{-2}{s+2}}_{} + \frac{3}{s+3}$$

$(-2)(-e^{-2t}u(-t))$

$(3)\, e^{-3t}u(t)$



Hence $x(t) = 3e^{-3t}u(t) + 2e^{-2t}u(-t)$

Find inverse laplace transform with ROC $= \{s \mid \mathrm{Re}(s) < -3\}$

$x(t) = 2e^{-2t}u(-t) - 3e^{-3t}u(-t)$

and with ROC $= \{s \mid \mathrm{Re}(s) > -2\}$

$x(t) = 3e^{-3t}u(t) - 2e^{-2t}u(t)$

LECTURE 27 and 28: 12th Sept.

$s \in ROC \Leftrightarrow \int_{-\infty}^{\infty} |x(t)| e^{-Re(s)t} dt < \infty$

Causality and Stability

$h(t) = 0$ for $t < 0$

- For **causality**, we know that the impulse response $h(t)$ is right-sided, therefore, ROC is to the right-side of the right-most pole
- For **stability** of such a system, we need $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
- This means that the $j\omega$ axis, i.e., $\{s \mid \mathrm{Re}(s) = 0\}$, be in the ROC of $H(s)$.
- A causal system with rational $H(s)$ is stable if and only if all the poles are to the left of the $j\omega$ axis.
- Determine causality and stability of

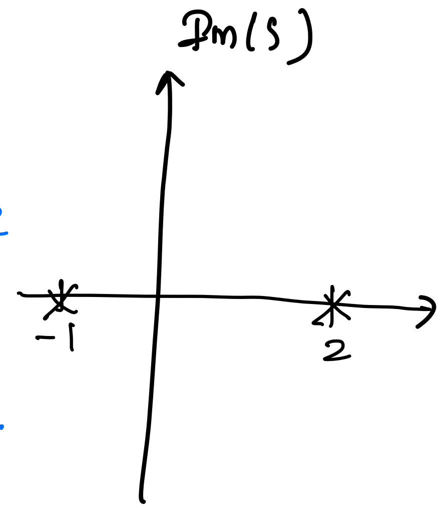
$$H(s) = \frac{s-1}{(s+1)(s-2)}.$$

poles $= \{-1, 2\}$

$ROC_1 = \{s \mid Re(s) \in (-1, 2)\} \Rightarrow$ not causal, stable

$ROC_2 = \{s \mid Re(s) > 2\} \Rightarrow$ causal, not stable

$ROC_3 = \{s \mid Re(s) < -1\} \Rightarrow$ neither causal nor stable.



Let $x(t) = x_1(t) - x_2(t)$, What is the ROC for $x(t)$?

$x_1(t) \longleftrightarrow \frac{1}{s+1}$, ROC: $Re(s) > -1$

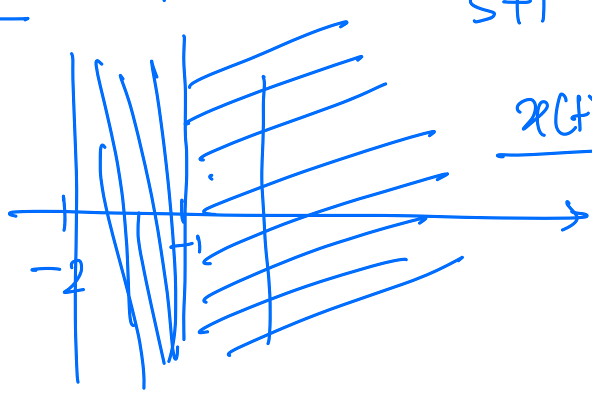
$x_2(t) \longleftrightarrow \frac{1}{(s+1)(s+2)}$, ROC: $Re(s) > -1$

$X(s) = X_1(s) - X_2(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{1}{s+2}$,

$x(t) = e^{-2t} u(t)$

ROC: $Re(s) > -2$

which contains intersection of ROCs of $X_1(s)$ & $X_2(s)$



Properties of Laplace Transform

- Linearity: $ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$, ROC contains the intersection.
- Time-Shift: $x(t - t_0) \longleftrightarrow e^{-st_0}X(s)$. ROC is not affected.

$$\int_{-\infty}^{\infty} e^{s_0 t} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-(s-s_0)t}\, dt = X(s-s_0)$$

let $\bar{s} \in \mathbb{C}$ that belongs to ROC of $x(t) \leftrightarrow X(s)$

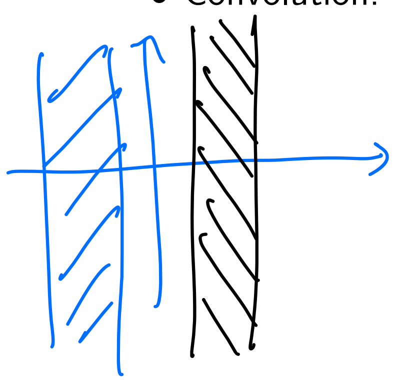
- Frequency Shift: $e^{s_0 t}x(t) \longleftrightarrow X(s - s_0)$. ROC now includes $\mathrm{Re}(s_0)$ + ROC of $x(t)$

we need to show $\bar{s} + \mathrm{Re}(s_0)$ belongs to ROC of $e^{s_0 t} x(t)$.

$$\int_{-\infty}^{\infty} \left|e^{s_0 t} x(t)\right| e^{-[\mathrm{Re}(\bar{s}) + \mathrm{Re}(s_0)]t}\, dt \le \int_{-\infty}^{\infty} |x(t)|\, \cancel{e^{\mathrm{Re}(s_0)t}}\, e^{-\mathrm{Re}(\bar{s})t}\, \cancel{e^{-\mathrm{Re}(s_0)t}}\, dt$$

$< \infty$ because $\bar{s} \in$ ROC of $x(t) \leftrightarrow X(s)$

- Convolution: $x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s)$. ROC contains the intersection.



Laplace transform of $u(t)$

$$\int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \frac{-1}{s} e^{-st}\Big|_0^{\infty} = \frac{1}{s} \quad \text{if } \mathrm{Re}(s) > 0.$$

let R be the ROC of $x(t)$.

Properties of Laplace Transform

- Differentiation: $\frac{dx(t)}{dt} \longleftrightarrow sX(s)$, ROC contains R

- Differentiation: $-tx(t) \longleftrightarrow \frac{dX(s)}{ds}$. , ROC = R

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \Rightarrow \frac{d}{ds}X(s) = \int_{-\infty}^{\infty} (-t x(t)) e^{-st} dt$

- Integration: $\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{1}{s}X(s)$, ROC containing $R \cap \{Re(s) > 0\}$

Note that $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau = \int_{-\infty}^{t} x(\tau) d\tau$

- Initial Value Theorem: For a causal signal $x(t)$ which does not contain an impulse at the origin, we have

$$x(0^+) = \lim_{s\to\infty} sX(s).$$

i.e., $x(t) = 0$ for $t < 0$,

- Final value theorem: For a causal signal $x(t)$ for which $\lim_{t\to\infty} x(t)$ is finite, we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

Laplace transform of some common signals.

① $x(t) = te^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^2}$, $\quad e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$

$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$ $\quad$ $\text{Re}(s) > -a$.

$-te^{-at}u(t) \longleftrightarrow \frac{d}{ds}\left(\frac{1}{s+a}\right)$

$= \frac{-1}{(s+a)^2}$

$X(s) = \frac{1}{(s+a)^n}$

② $x(t) = \delta(t)$, $\quad X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1$,

ROC $= \mathbb{C}$

③ $x(t) = u(t)$, $\quad X(s) = \frac{1}{s}$, $\quad$ ROC $= \{s \mid \text{Re}(s) > 0\}$

$x(t) = \frac{t^{n-1}}{(n-1)!}u(t)$, $\quad x(t) = tu(t) \longleftrightarrow X(s) = \frac{1}{s^2}$,

ROC $= \{s \mid \text{Re}(s) > 0\}$

$X(s) = \frac{1}{s^n}$.

④ $x(t) = \cos(\omega_0 t)u(t)$

$x(t) = e^{-\alpha t}\cos(\omega_0 t)u(t)$, $\alpha \in \mathbb{R}$

$x(t) = \sin(\omega_0 t)u(t)$

$x(t) = \cos(\omega_0 t)$

$x(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

$X(s) = \frac{1}{2} \cdot \frac{1}{s - j\omega}$,

$+ \frac{1}{2} \frac{1}{s + j\omega}$,

ROC: $\{s \mid \text{Re}(s) > 0\}$

$= \frac{s}{s^2 + \omega^2}$,

$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$X(s) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-(s - j\omega_0)t}\, dt = \frac{-1}{s - j\omega_0} e^{-(s-j\omega_0)t}\Big|_{-\infty}^{\infty}$

↳ does not converge for any s

Laplace transform does not exist $\Leftarrow$

Observe the Laplace and Fourier Transforms

① $x(t) = \cos(\omega_0 t)$ → Laplace transform does not exist

→ $\mathcal{F}(x(t)) = \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$.

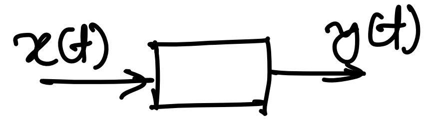
② $x(t) = e^{j\omega_0 t}$

→ similar property holds for other periodic signals as well.

Can we claim that Fourier transform $= X(s)\big|_{s=j\omega}$?

→ only when ROC of Laplace transform includes the imaginary axis.

Finding Impulse Response of Differential Equations



- consider the example

$$\frac{d}{dt}y(t) + 3y(t) = x(t)$$

can we uniquely determine its impulse response using Laplace transform?

applying Laplace transform,

$$sY(s) + 3Y(s) = X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}.$$

However, the above approach does not provide information about ROC.

If we have additional information about causality (or stability), we can determine ROC and find $h(t)$.

For a LTI system, suppose $x(t) = e^{-3t}u(t)$ applied as input results in output $y(t) = [e^{-t} - e^{-2t}]u(t)$

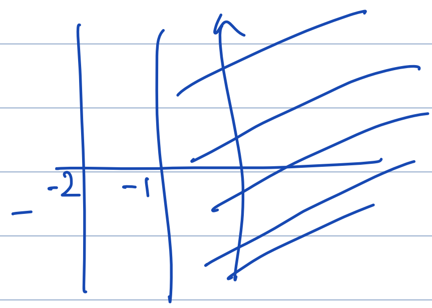
Determine if the system is stable and causal.

$X(s) = \frac{1}{s+3}$, ROC: $\{s \mid Re(s) > -3\}$

$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$, ROC: $\{s \mid Re(s) > -1\}$

$H(s) = \frac{Y(s)}{X(s)} = (s+3) \cdot \frac{(s+2-(s+1))}{(s+1)(s+2)}$

$= \frac{s+3}{(s+1)(s+2)}$, ROC: $\{s \mid Re(s) > -1\}$



$\Rightarrow$ system is both causal and stable.

Unilateral Laplace Transform

$$X_u(s) = \int_{0^-}^{\infty} x(t) e^{-st}\, dt = \mathcal{U}\mathcal{L}(x(t))$$, ROC is a right half plane.

Ex: determine $X_u(s)$ for $x(t) = \delta(t) + e^{t} u(t)$

$$\int_{0^-}^{\infty} (\delta(t) + e^{t}) e^{-st}\, dt = 1 + \int_{0^-}^{\infty} e^{-(s-1)t}\, dt$$

$$= 1 + \frac{1}{s-1}, \quad Re(s) > 1$$

Ex: determine $x(t)$ for which $X_u(s) = \dfrac{1}{(s+1)(s+2)}$.

It is implicit that ROC is the region to the right of the right most pole.

$$X_u(s) = \frac{1}{s+1} - \frac{1}{s+2}, \quad ROC = \{s \mid Re(s) > -1\}$$

$$x(t) = (e^{-t} - e^{-2t}) u(t)$$

Differentiation Property

$$\int_{0^-}^{\infty} \frac{d}{dt}x(t)\, e^{-st}\, dt = s X_u(s) - x(0^-) \longleftrightarrow \frac{d}{dt}x(t)$$

By taking integration by parts, we obtain

$$\underbrace{x(t)e^{-st}\Big|_{0^-}^{\infty}} + \int_{0^-}^{\infty} s\, x(t)\, e^{-st} dt = -x(0^-) + s\int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$= s X_u(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \longleftrightarrow s^2 X_u(s) - s\,x(0^-) - \frac{d}{dt}x(0^-)$$

Ex: consider a LTI system whose input $x(t)$ and output $y(t)$ are governed by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

let $x(t) = 2u(t)$, $y(0^-) = 3$, $y'(0^-) = -5$.

Determine the output $y(t)$, and state which part of the output is due to input and initial conditions.

applying unilateral laplace transform, we obtain

$$s^2 Y_u(s) - s\,y(0^-) - y'(0^-) + 3\left[s Y_u(s) - y(0^-)\right]$$
$$+ 2Y_u(s) = X_u(s) = \frac{2}{s}$$

$$\Rightarrow (s^2+3s+2)Y_u(s) - (3+s)\,y(0^-) - y'(0^-) = X_u(s)$$

$$\Rightarrow Y_u(s) = \underbrace{\frac{X_u(s)}{s^2+3s+2}}_{\text{depends on } x(t)} + \underbrace{\frac{(3+s)\,y(0^-) + y'(0^-)}{(s^2+3s+2)}}_{\text{does not depend on } x(t)}$$

$$= \frac{2}{s(s+2)(s+1)} + \frac{(3+s)3-5}{(s+2)(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

find $y(t)$

Pole-zero cancellation and differential equations

$$G(s) = \frac{1}{s+1}, \quad \tilde{G}(s) = \frac{s+2}{(s+1)(s+2)}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}, \quad \dot{x} = f(x)$$

$$\frac{d^2y(t)}{dt^2} + \;--$$

$$x_1 = y(t)$$

$$x_2 = \frac{d}{dt}(y(t))$$

Practice problems from textbook

Chapter 6 : 6.9, 6.10, 6.11, 6.12

Chapter 9 : 9.2, 9.4, 9.7, 9.8, 9.16, 9.13, 9.22, 9.26
9.31, 9.33

Chapter 4 : 4.1, 4.2, 4.31, 4.6, 4.17