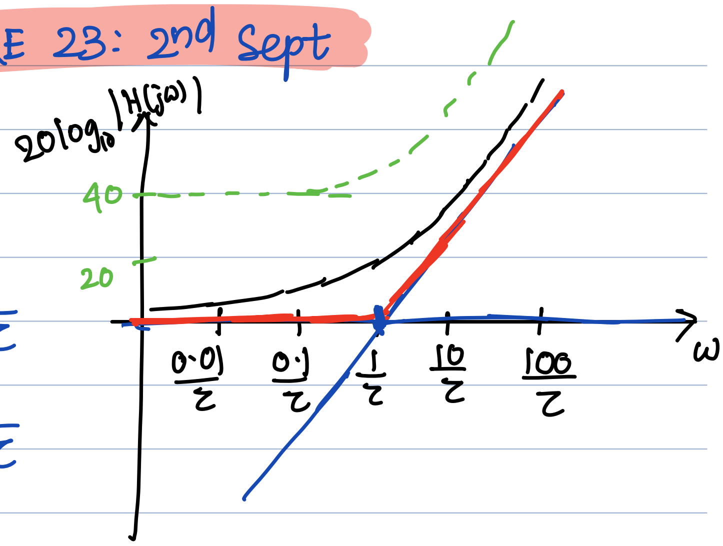
LECTURE 23: 2nd Sept

$$H(j\omega) = 1 + j\omega\tau$$

$$|H(j\omega)| = \sqrt{1+\omega^2\tau^2}$$

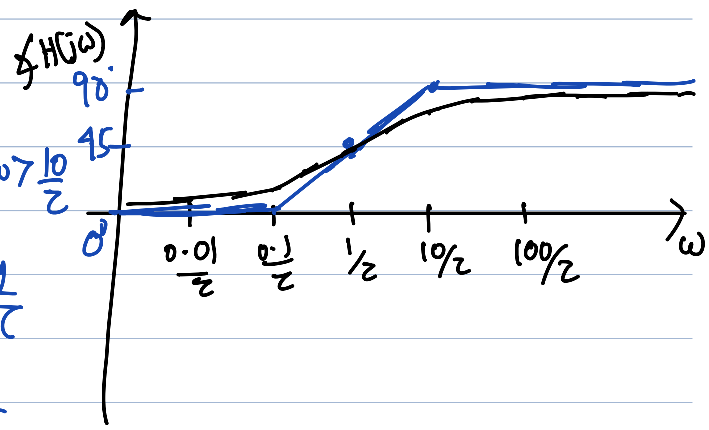
$$= \begin{cases} 1 , & \omega \ll \frac{1}{\tau} \\ \omega\tau , & \omega \gg \frac{1}{\tau} \end{cases}$$



$20\log_{10}|H(j\omega)|$

$$= \begin{cases} 0 & \omega < 0.1/\tau \\ 20\log_{10}\omega + 20\log_{10}\tau , & \omega > \frac{10}{\tau} \end{cases}$$

$$\measuredangle H(j\omega) = \begin{cases} 0^\circ & \omega \ll \frac{1}{\tau} \\ 90^\circ & \omega \gg \frac{1}{\tau} \end{cases}$$



draw Bode plot of $\underline{H(j\omega) = 100(1+j\omega\tau)}$

$$\underline{20\log_{10}|H(j\omega)|} = \underbrace{20\log_{10}100}_{40} + 20\log_{10}|1+j\omega\tau|$$

$$\measuredangle H(j\omega) = \measuredangle(1+j\omega\tau)$$

<u>More generally:</u> $H(j\omega) = \dfrac{(j\omega+c_1)(j\omega+c_2)\cdots}{(j\omega+\alpha_1)(j\omega+\alpha_2)\cdots}$

$$20\log_{10}|H(j\omega)| = 20\log_{10}|j\omega+c_1| + 20\log_{10}|j\omega+c_2| + \cdots - 20\log_{10}|j\omega+\alpha_1| - 20\log_{10}|j\omega+\alpha_2| \cdots$$

$$\angle H(j\omega) = \angle j\omega + c_1 + \angle j\omega + c_2 + \cdots - \angle j\omega + \alpha_1 - \angle j\omega + \alpha_2 \cdots$$

Sketch asymptotic Bode plot of $H(j\omega) = \dfrac{1+j\omega}{10+j\omega}$

$$H(j\omega) = \frac{1+j\omega}{1+j\omega\left(\frac{1}{10}\right)} \cdot \frac{1}{10}$$

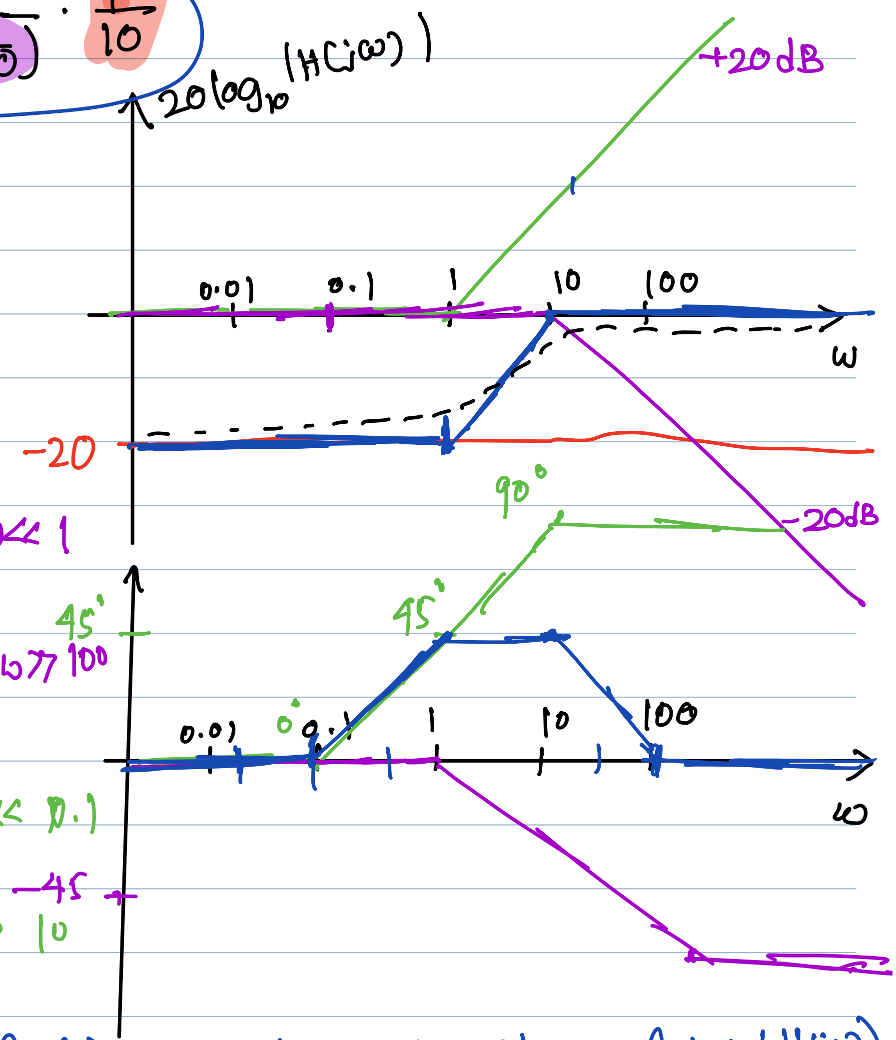
$20\log_{10}\left(\frac{1}{10}\right) = -20$

$20\log_{10}|1+j\omega|$

$20\log_{10}\left|\dfrac{1}{1+j\omega(0.1)}\right|$

$$\angle(1+j\omega 0.1)^{-1} = \begin{cases} 0^\circ & \omega \ll 1 \\ -90^\circ, & \omega \gg 100 \end{cases}$$

$$\angle(1+j\omega) = \begin{cases} 0^\circ & \omega \ll 0.1 \\ 90^\circ & \omega \gg 10 \end{cases}$$



If $x(t) = \cos(\omega t)$, $y(t) = |H(j\omega)|\cos(\omega t + \angle H(j\omega))$

Note: The significance of Bode plot does not lie in the fact that we can draw Bode plot given $H(j\omega)$. Rather, we can infer $H(j\omega)$ given the (asymptotic) Bode plot.

often, Bode plots are determined experimentally by applying $x(t) = \cos(\omega t)$ for a wide range of ω, and from the output, we determine $\underline{|H(j\omega)|}$ and $\underline{\measuredangle H(j\omega)}$ at those frequencies.

<u>First Order system</u> : $\tau \frac{d}{dt} y(t) + y(t) = x(t)$

applying Fourier transform, we obtain

$$\tau j\omega\, Y(j\omega) + Y(j\omega) = X(j\omega)$$

$$\Rightarrow \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1+j\omega\tau} = \frac{1}{\tau}\left(\frac{1}{1/\tau + j\omega}\right)$$

- <u>ex</u>: RLC circuit with $\tau = RC$.
- we have already seen how Bode plot looks like.

<u>Impulse response</u>: $h(t) = \mathcal{F}^{-1}[H(j\omega)] = \frac{1}{\tau} e^{-t/\tau} u(t)$

If τ is larger, decay is slower.

τ: time-constant of the system.

<u>Step Response</u>: determine $y(t)$ when input is $u(t)$

$$Y(j\omega) = \frac{1}{1+j\omega\tau} \cdot \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{1}{j\omega} \cdot \frac{1}{1+j\omega\tau} + \pi\,\delta(\omega)\frac{1}{1+j\omega\tau}$$

$$= \frac{A}{j\omega} + \frac{B}{1+j\omega\tau} + \pi\,\delta(\omega)\left[\frac{1-j\omega\tau}{(1+j\omega\tau)(1-j\omega\tau)}\right]$$

$$= \left(\frac{1}{j\omega}\right) - \left[\frac{\tau}{1+j\omega\tau}\right] + \pi\,\delta(\omega)\left[\frac{1}{1+\omega^2\tau^2} - j\,\frac{\omega\tau}{1+\omega^2\tau^2}\right]$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{j\omega} + \pi\,\delta(\omega)\left(\frac{1}{1+\omega^2\tau^2}\right) - j\pi\,\frac{\delta(\omega)\,\omega\tau}{1+\omega^2\tau^2}\right]e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{j\omega} + \pi\,\delta(\omega)\right)e^{j\omega t}\,d\omega$$

$$= u(t)$$

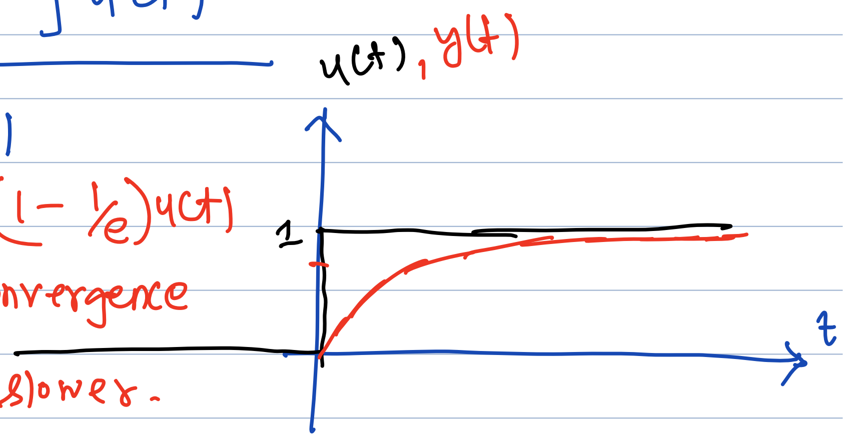
$$y(t) = u(t) - \tau\,\mathcal{F}^{-1}\left(\frac{1}{1+j\omega\tau}\right)$$

$$= \left[1 - e^{-t/\tau}\right]u(t)$$

as $t \to \infty$, $y(t) \to 1$

at $t = \tau$, $y(t) = \left(1 - \frac{1}{e}\right)u(t)$

– If τ increases, convergence to steady-state is slower.



LECTURE 24: 4th Sept.

A Second order system is represented as a ODE as

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{d}{dt}y(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$$

$$x(t) \rightarrow \boxed{\quad\quad} \rightarrow y(t)$$

ω_n: natural frequency.
ζ: damping parameter
(zeta)

Applying Fourier transform, we obtain

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n(j\omega)Y(j\omega) + \omega_n^2 Y(j\omega) = \omega_n^2 X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{A}{j\omega + C_1} + \frac{B}{j\omega + C_2}$$

$h(t) = \left[Ae^{-C_1t} + Be^{-C_2t}\right]u(t)$ when $C_1 \neq C_2$

$$= \frac{(A+B)j\omega + AC_2 + BC_1}{(j\omega)^2 + (C_1+C_2)j\omega + C_1C_2}$$

comparing the coefficients, we obtain:

$$\boxed{A+B=0}, \quad \boxed{AC_2 + BC_1 = \omega_n^2}$$

$$C_1C_2 = \omega_n^2, \quad C_1 + C_2 = 2\zeta\omega_n$$

$$\Rightarrow \boxed{C_2 = 2\zeta\omega_n - C_1}$$

$$\Rightarrow C_1\left[2\zeta\omega_n - C_1\right] = \omega_n^2$$

$$\Rightarrow C_1^2 - 2\zeta\omega_n C_1 + \omega_n^2 = 0$$

$$\Rightarrow c_1 = \zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$$

$$\begin{cases} c_1 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} > 0 \\ c_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1} > 0 \end{cases}$$

$$Ae^{-c_1 t} + Be^{-c_2 t}$$

$$e^{-\zeta\omega_n t}[\ ]$$

We have the following three possibilities.

(i) $\zeta = 1 \Rightarrow c_1 = c_2 = \zeta\omega_n$

(ii) $\zeta > 1 \Rightarrow c_1$ and c_2 are both real

(iii) $\zeta < 1 \Rightarrow$ $c_1 = \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

$c_2 = \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$

Now, let us solve for A and B.

We know: $A + B = 0 \Rightarrow B = -A$

$$Ac_2 + Bc_1 = \omega_n^2$$

$$\Rightarrow A(c_2 - c_1) = \omega_n^2$$

$$\Rightarrow A = \frac{\omega_n^2}{-2\omega_n\sqrt{\zeta^2-1}} \quad \text{when } \zeta \neq 1$$

Now, let us determine $h(t) = \mathcal{F}^{-1}[H(j\omega)]$.
for all three cases.

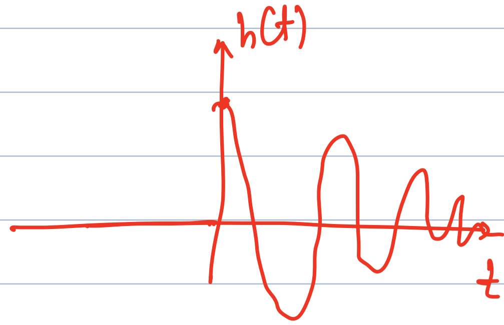
Case 1: $\zeta > 1$

$$h(t) = Be^{-\zeta\omega_n t} \cdot u(t) \cdot \left[e^{\sqrt{\zeta^2-1}\,\omega_n t} - e^{-\sqrt{\zeta^2-1}\,\omega_n t}\right]$$

$$B = \frac{\omega_n}{2\sqrt{\zeta^2-1}}$$

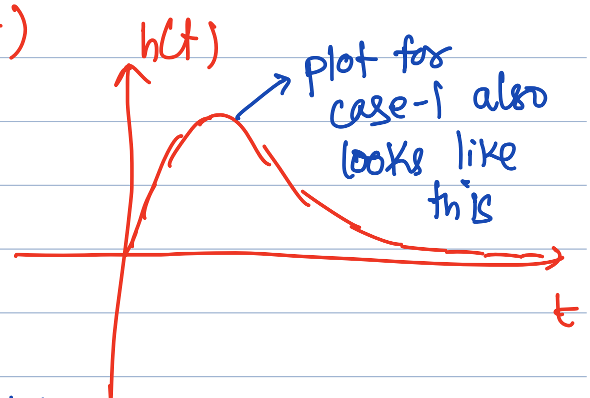
Case 2: $\zeta < 1$

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \cdot \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \cdot u(t)$$



Case 3: $\zeta = 1$

$$h(t) = \underline{\omega_n^2 t}\; \underbrace{e^{-\zeta\omega_n t}}_{\text{will dominate the linearly increasing function } t.} \cdot u(t)$$



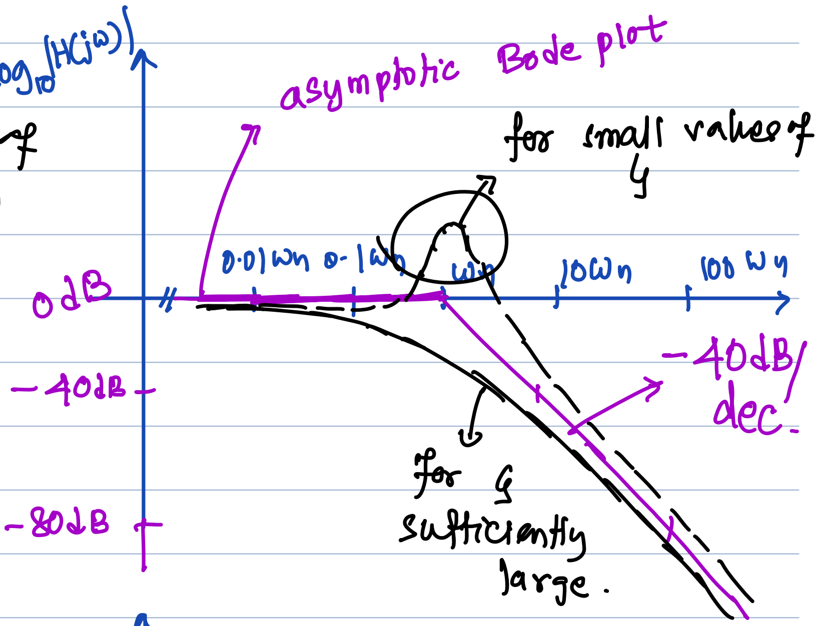
Finally, let us sketch the Bode plot

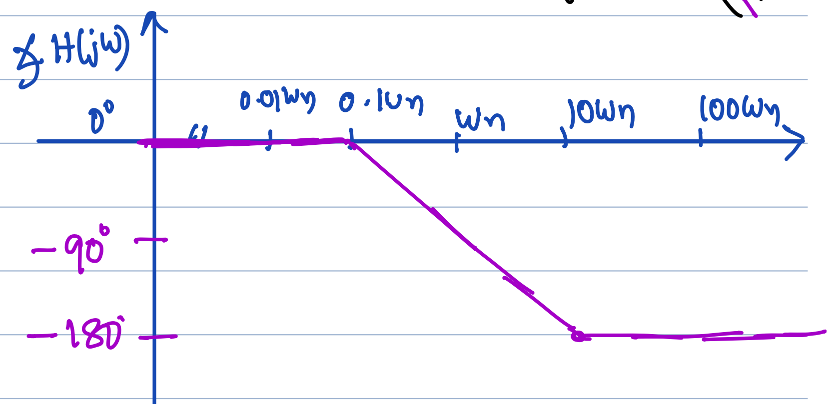
$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + \underline{2\zeta\omega_n \omega j} + \omega_n^2} \approx -\frac{\omega_n^2}{\omega^2}$$

$$|H(j\omega)| = \begin{cases} 1, & \omega \ll \omega_n \\ \dfrac{\omega_n^2}{\omega^2}, & \omega \gg \omega_n \end{cases}, \quad \angle H(j\omega) = \begin{cases} 0^\circ, & \omega \ll \omega_n \\ -180^\circ, & \omega \gg \omega_n \\ ?, & \omega = \omega_n \end{cases}$$

$$20\log_{10}|H(j\omega)| = \begin{cases} 0, & \omega \ll \omega_n \\ \underbrace{-40\log_{10}\omega + 40\log_{10}\omega_n}_{=0 \text{ if } \omega = \omega_n}, & \omega \gg \omega_n \end{cases}$$

(HW) determine the range of ζ for which we will have an overshoot





Note the difference in the asymptotic Bode plot for first and second order systems for large ω.

<u>slope of magnitude plot</u>: -20 dB/dec for 1st order
-40 dB/dec for 2nd order

<u>phase plot</u>: converges to -90° for 1st
" -180° for 2nd