LECTURE-19 : 27th August

Partial Fraction General Case

- For convenience: consider $j\omega$ as a variable v:

$$H(v) = \frac{b_{n-1}v^{n-1} + b_{n-2}v^{n-2} + \cdots + b_1 v + b_0}{v^n + a_{n-1}v^{n-1} + a_{n-2}v^{n-2} + \cdots + a_1 v + a_0} = \frac{b_{n-1}v^{n-1} + b_{n-2}v^{n-2} + \cdots + b_1 v + b_0}{(v-p_1)^{\sigma_1}(v-p_2)^{\sigma_2}\cdots(v-p_r)^{\sigma_r}}$$

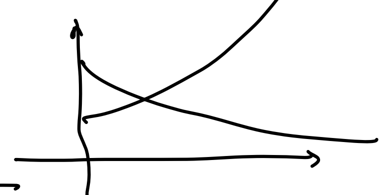
- We write this as a sum of simpler polynomial ratios as

$$\begin{aligned} H(v) &= \frac{A_{11}}{(v-p_1)} + \frac{A_{12}}{(v-p_1)^2} + \cdots + \frac{A_{1\sigma_1}}{(v-p_1)^{\sigma_1}} \\ &+ \frac{A_{21}}{(v-p_2)} + \frac{A_{22}}{(v-p_2)^2} + \cdots + \frac{A_{2\sigma_2}}{(v-p_2)^{\sigma_2}} \\ &\cdots \\ &+ \frac{A_{r1}}{(v-p_r)} + \frac{A_{r2}}{(v-p_r)^2} + \cdots + \frac{A_{r\sigma_r}}{(v-p_r)^{\sigma_r}} \end{aligned}$$

- We can inverse CTFT each term using $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^n}$ for $a > 0$.

for n=2 : $t e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$

$$j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) = j\,\frac{-1}{(a+j\omega)^2}\cdot j = \frac{1}{(a+j\omega)^2}$$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} x(t)\,(-jt)\,e^{-j\omega t}\,dt$$

$$j\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} (t\,x(t))\,e^{-j\omega t}\,dt$$

Recall

$x(t) \longleftrightarrow X(j\omega)$

$t\,x(t) \longleftrightarrow j\frac{d}{d\omega}X(j\omega)$

$t^2 x(t) \longleftrightarrow (j)^2 \frac{d^2}{d\omega^2}X(j\omega)$

$$\mathcal{F}\left[t^2 e^{-at}u(t)\right] = j\frac{d}{d\omega}\left(\frac{1}{(a+j\omega)^2}\right) = j\,\frac{-2}{(a+j\omega)^3}\,j = \frac{2}{(a+j\omega)^3}$$

$\frac{d}{dt} y(t) \longleftrightarrow j\omega Y(j\omega)$

$\frac{d^2 y(t)}{dt^2} \longleftrightarrow (j\omega)^2 Y(j\omega) = -\omega^2 Y(j\omega)$

$y(t) \longleftrightarrow Y(j\omega)$

$x(t) \longleftrightarrow X(j\omega)$

Example

Calculate the output for the input $x(t) = e^{-t}u(t)$, for a system with the input-output relationship:

$$\frac{d^2y(t)}{dt^2} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

apply Fourier Transform to obtain

$$-\omega^2 Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$$

$$\Rightarrow Y(j\omega)\left[3-\omega^2\right] = (2+j\omega)X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2+j\omega}{3+(j\omega)^2}$$

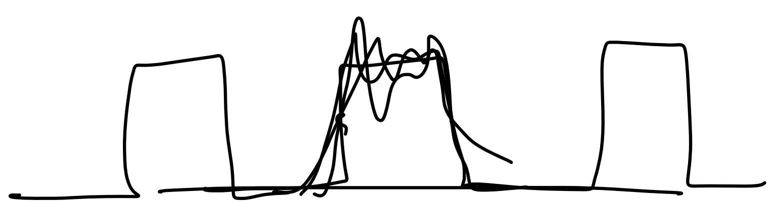
$$= \frac{2+j\omega}{(j\omega+j\sqrt{3})(j\omega-j\sqrt{3})}$$

$$= \frac{A_1}{j\omega+j\sqrt{3}} + \frac{A_2}{j\omega-j\sqrt{3}}$$

$$\Rightarrow h(t) = \left(A_1 e^{-j\sqrt{3}t} + A_2 e^{j\sqrt{3}t}\right)u(t).$$

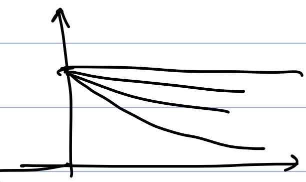
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t_0}d\omega$$

$$= \frac{x(t_0^+) + x(t_0^-)}{2}$$



Fourier transform of unit step function

$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t) = \lim_{a \to 0} e^{-at} u(t)$



$\mathcal{F}[e^{-at} u(t)] = \frac{1}{a + j\omega}$

$\lim_{a \to 0} \mathcal{F}[e^{-at} u(t)] = \lim_{a \to 0} \frac{1}{a + j\omega}$ → we cannot set $a = 0$ and conclude that limit is $\frac{1}{j\omega}$.

$= \lim_{a \to 0} \frac{a - j\omega}{(a + j\omega)(a - j\omega)}$

$= \lim_{a \to 0} \frac{a - j\omega}{a^2 + \omega^2}$

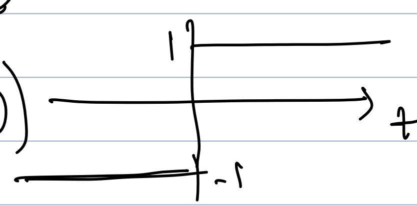
$= \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2} - j \boxed{\lim_{a \to 0} \frac{\omega}{a^2 + \omega^2}} \right] = (\ \cdot\ ) - j \frac{\omega}{\omega^2} + \frac{1}{j\omega}$

$\boxed{\begin{array}{l} \text{if } \omega \neq 0 : \ 0 \\ \text{if } \omega = 0 : \ \infty \end{array}}$

$\int_{-\infty}^{\infty} \frac{a}{a^2 + \omega^2} \, d\omega = \tan^{-1}\left(\frac{\omega}{a}\right) \Big|_{-\infty}^{\infty} = \pi$

$= \pi \delta(\omega) + \frac{1}{j\omega}$ $\quad u(t) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(t)$

$\operatorname{sgn}(t) = \lim_{a \to 0} \left(\boxed{e^{-at} u(t) - e^{-at} u(-t)} \right)$



$\operatorname{sgn}(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$

$$u(t) = \frac{1}{2} + \frac{1}{2} \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

Example

Suppose a causal stable continuous-time LTI system has frequence response

$$H(j\omega) = \frac{4 + j\omega}{6 - \omega^2 + j5\omega}.$$

Determine its impulse response in time-domain.

$$= \frac{4 + j\omega}{(2 + j\omega)(3 + j\omega)}$$

$$= \frac{2}{2 + j\omega} + \frac{-1}{3 + j\omega}$$

$$h(t) = 2e^{-2t} u(t) - e^{-3t} u(t)$$

LECTURE 20 : 28th Aug.

PRACTICE PROBLEMS

Chapter 1 : 11, 13, 14, 20, 26, 27, 28, 31
32, 34, 40, 41, 42, 43

Chapter 2 : 14, 15, 16, 19, 28, 31, 38, 44, 46, 48

Chapter 3 : 4, 13, 18, 19, 21, 26, 40, 44, 48,
63, 64

Chapter 4 : 24, 26, 30, 31, 44

(4.11) $y(t) = x(t) \circledast h(t)$

$x(t) \longleftrightarrow X(j\omega)$

$h(t) \longleftrightarrow H(j\omega)$

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$Y(j\omega) = X(j\omega)H(j\omega)$

Given : $g(t) = x(3t) \circledast h(3t)$

can we express $g(t) = A\, y(Bt)$, for some scalars A & B ?

$x(3t) \longleftrightarrow \frac{1}{3}X\left(\frac{j\omega}{3}\right)$

$h(3t) \longleftrightarrow \frac{1}{3}H(j\omega/3)$

$\Rightarrow G(j\omega) = \frac{1}{9}\, \underbrace{X(j\omega/3)\, H(j\omega/3)} = \frac{1}{3}\left[\frac{1}{3}X(j\omega/3)\, H(j\omega/3)\right]$

$= \frac{1}{3}\left[\underline{\frac{1}{3}Y\left(j\omega/3\right)}\right]$

$\Rightarrow g(t) = \frac{1}{3}\, \underline{y(3t)}$

4.12

$$e^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2}$$

a) Find FT of $te^{-|t|}$

b) using duality, determine FT of $\boxed{\frac{4t}{(1+t^2)^2}}$.

Solution:

(a) $$\mathcal{F}\left[te^{-|t|}\right] = j\frac{d}{d\omega}\left[\frac{2}{1+\omega^2}\right] = j\,\frac{-2}{(1+\omega^2)^2}\cdot 2\omega$$

$$= j\,\frac{-4\omega}{(1+\omega^2)^2}$$

$$= -4j\,\frac{\omega}{(1+\omega^2)^2}$$

$$\mathcal{F}[x(t)] = X(j\omega)$$

$$\mathcal{F}[X(jt)] = 2\pi x(-\omega)$$

$$g(t) = -4j\,\frac{t}{(1+t^2)^2}$$

$$= (-j)\,\frac{4t}{(1+t^2)^2}$$

$$\mathcal{F}[g(t)] = 2\pi(-\omega)e^{-|\omega|}$$

$$\Rightarrow \frac{1}{(-j)}\mathcal{F}[g(t)] = \frac{2\pi(+\omega)e^{-|\omega|}}{(+j)} = \frac{2\pi\omega}{j}e^{-|\omega|}$$

4.19

$$H(j\omega) = \frac{1}{j\omega+3}$$

$$x(t) \longrightarrow \boxed{} \longrightarrow y(t)$$

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t) \Rightarrow Y(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega+4}$$

Determine $x(t)$.

solution: $X(j\omega)\,H(j\omega) = Y(j\omega)$

$$\Rightarrow X(j\omega) = \frac{\left(\frac{1}{j\omega+3} - \frac{1}{j\omega+4}\right)}{\frac{1}{j\omega+3}}$$

$$= 1 - \frac{j\omega+3}{j\omega+4} = \frac{(j\omega+4)-(j\omega+3)}{j\omega+4}$$

$$= \frac{1}{j\omega+4}$$

$$x(t) = e^{-4t}u(t).$$

4.15 i) $x(t)$ is real $\quad x(t) \longleftrightarrow X(j\omega)$

ii) $x(t) = 0$ for $t \le 0$

iii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} Re[X(j\omega)]\, e^{j\omega t}\, d\omega = |t|e^{-|t|}$.

$X(-j\omega) = X(j\omega)^*$

Determine $x(t)$.

Solution:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(-j\bar{\omega})e^{j\bar{\omega} t}\, d\bar{\omega} \qquad \bar{\omega} = -\omega$$

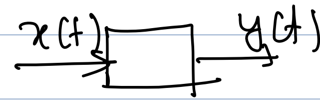
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)^* e^{j\omega t}\, d\omega$$

$$|t|e^{-|t|} = \frac{x(t)+x(-t)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X(j\omega)+X(j\omega)^*}{2} e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = 2t\,e^{-|t|} \quad \text{for } t > 0$$

4.44

$$\frac{d}{dt}y(t) + 10y(t) = \int_{-\infty}^{\infty} x(\tau)z(t-\tau)d\tau - x(t),$$



$$z(t) = e^{-t}u(t) + 3\delta(t) \Rightarrow Z(j\omega) = \frac{1}{j\omega+1} + 3$$

Determine impulse response of the above system.

applying Fourier Transform, we obtain

$$j\omega Y(j\omega) + 10Y(j\omega) = X(j\omega)Z(j\omega) - X(j\omega).$$

$$= X(j\omega)\left[Z(j\omega) - 1\right]$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega)-1}{j\omega+10} = \frac{\frac{1}{j\omega+1} + 2}{j\omega+10}$$

$$= \frac{2j\omega+3}{(j\omega+1)(j\omega+10)} = \frac{A_1}{j\omega+1} + \frac{A_2}{j\omega+10}$$

$$= \frac{1}{9}\cdot\frac{1}{j\omega+1} + \frac{17}{9}\cdot\frac{1}{j\omega+10}$$

$$h(t) = \left(\frac{1}{9}e^{-t} + \frac{17}{9}e^{-10t}\right)u(t).$$

4.36

$$x(t) = \left(e^{-t} + e^{-3t}\right)u(t)$$

$$y(t) = \left(2e^{-t} - 2e^{-4t}\right)u(t)$$

find a differential equation that relates $x(t)$ & $y(t)$.

solution:

$$Y(j\omega) = \frac{2}{1+j\omega} - \frac{2}{4+j\omega} = \frac{6}{(1+j\omega)(4+j\omega)}$$

$$X(j\omega) = \frac{1}{1+j\omega} + \frac{1}{3+j\omega} = \frac{2j\omega+4}{(1+j\omega)(3+j\omega)}$$

$$H(j\omega) = \frac{6}{\cancel{(1+j\omega)}(4+j\omega)} \cdot \frac{\cancel{(1+j\omega)}(3+j\omega)}{2j\omega+4}$$

$$= \frac{3(3+j\omega)}{(4+j\omega)(j\omega+2)}$$

$$= \frac{9+j3\omega}{8-\omega^2+6j\omega} \doteq \frac{Y(j\omega)}{X(j\omega)}$$

$$9x(t) + 3\frac{d}{dt}x(t) = 8y(t) + \frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t).$$

LECTURE 21 & 22: 29th Aug.

Suppose we apply input

$$x(t) = \cos(\omega_0 t)$$

$$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$y(t) = \frac{1}{2}e^{j\omega_0 t}H(j\omega_0) + \frac{1}{2}e^{-j\omega_0 t}H(-j\omega_0)$$

$$= \frac{1}{2}|H(j\omega_0)|\, e^{j\measuredangle H(j\omega_0)}\, e^{j\omega_0 t}$$

$$+ \frac{1}{2}|H(j\omega_0)|\, e^{-j\measuredangle H(j\omega_0)}\, e^{-j\omega_0 t}$$

$$= \frac{1}{2}|H(j\omega_0)|\left[e^{j(\omega_0 t + \measuredangle H(j\omega_0))} + e^{-j(\omega_0 t + \measuredangle H(j\omega_0))}\right]$$

$$= |H(j\omega_0)| \cos(\omega_0 t + \underbrace{\measuredangle H(j\omega_0)}_{\text{phase}})$$

$$e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

$$\mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

$$= H(j\omega)$$

— transfer function

In general, $H(j\omega)$ is complex.

$$H(j\omega) = |H(j\omega)|\, e^{j\measuredangle H(j\omega)}$$

Now, if $h(t)$ is real-valued,

$$H(-j\omega) = H(j\omega)^*$$

$$= |H(j\omega)|\, e^{-j\measuredangle H(j\omega)}$$

Recall:

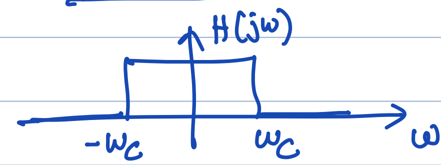
$$(re^{j\theta})^* = re^{-j\theta}$$

Note: when input is $\cos(\omega_0 t)$, output is of the form
$\underline{A\cos(\omega_0 t+\phi)}$, where $A = |H(j\omega_0)|$ ✓
$\phi = \measuredangle H(j\omega_0)$ ✓

Ideal Low-pass filter ✓

LTI system with

$$H(j\omega) = \begin{cases} 1 & \text{if } |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$



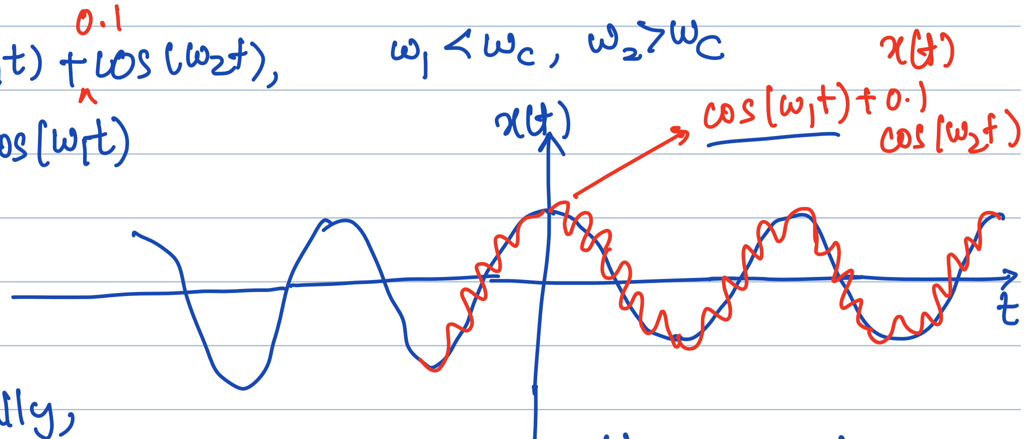
ω_c: cut-off frequency.

If $\underline{x(t)} = \underline{\cos(\bar{\omega}t)}$, with $\underline{\bar{\omega}} > \underline{\omega_c}$,
then $y(t) = 0$

$H(j\bar{\omega}) = 0$

if $\bar{\omega} \in [-\omega_c, \omega_c]$, $\underline{y(t) = x(t)}$.

If $x(t) = \cos(\omega_1 t) + 0.1\cos(\omega_2 t)$, $\omega_1 < \omega_c$, $\omega_2 > \omega_c$
then $y(t) = \cos(\omega_1 t)$



More generally,

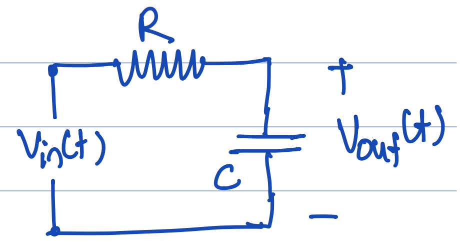
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,\underline{H(j\omega)}\,e^{j\omega t}\,d\omega = \underline{\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} X(j\omega)\,e^{j\omega t}\,d\omega}$$

Approximating Ideal Low Pass Filter with RC circuit

Applying KVL, we obtain

$$V_{in}(t) = R\,i(t) + V_{out}(t)$$

$$= RC\frac{d}{dt}V_{out}(t) + V_{out}(t)$$



applying Fourier transform, we obtain:

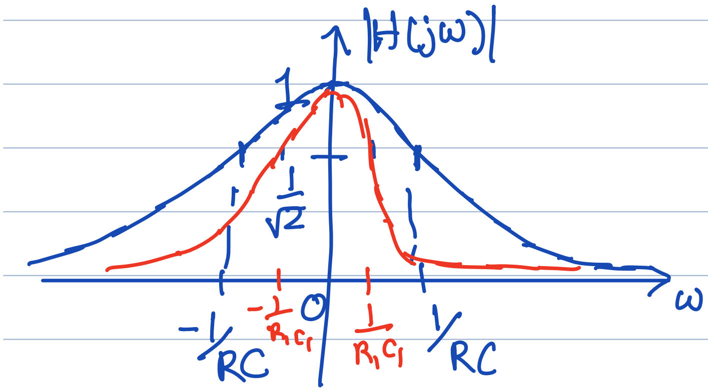
$$V_{in}(j\omega) = [RC\, j\omega + 1]\, V_{out}(j\omega)$$

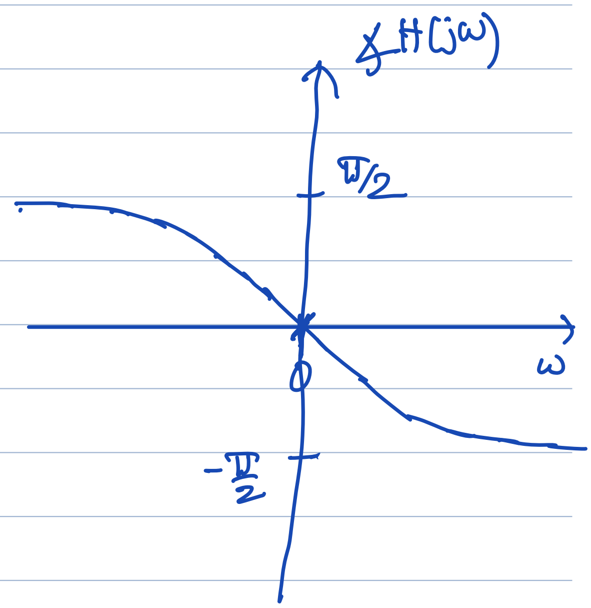
$$\Rightarrow \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = H(j\omega) = \frac{1}{1 + j\omega RC}$$

: transfer function of a first-order system as denominator has 1 root

Let us sketch $|H(j\omega)|$ and $\measuredangle H(j\omega)$ as a function of ω.

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}, \quad \measuredangle H(j\omega) = \underline{\tan^{-1}(-\omega RC)}$$

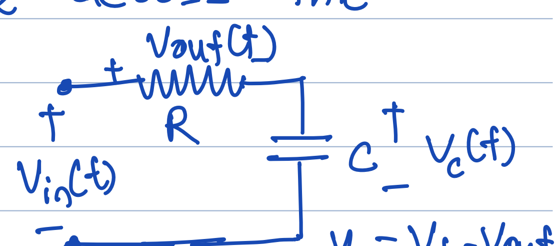




– approximates a low-pass filter transfer function.

Now we consider the case where output is defined to be the voltage across the resistor.

$$\cancel{V_{in}(t)} = RC\frac{dV_C}{dt} + V_C$$



$V_C = V_{in} - V_{out}$

$$= RC\frac{dV_{in}}{dt} - RC\frac{dV_{out}}{dt} + \cancel{V_{in}} - V_{out}$$

$$\Rightarrow V_{out} + RC\frac{d}{dt}V_{out} = RC\frac{d}{dt}V_{in}$$

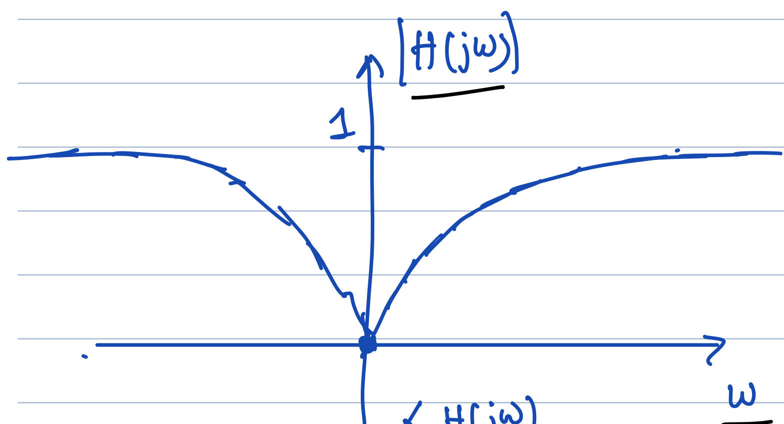
$$\Rightarrow V_{out}(j\omega) + j\omega RC\, V_{out}(j\omega) = j\omega RC\, V_{in}(j\omega).$$

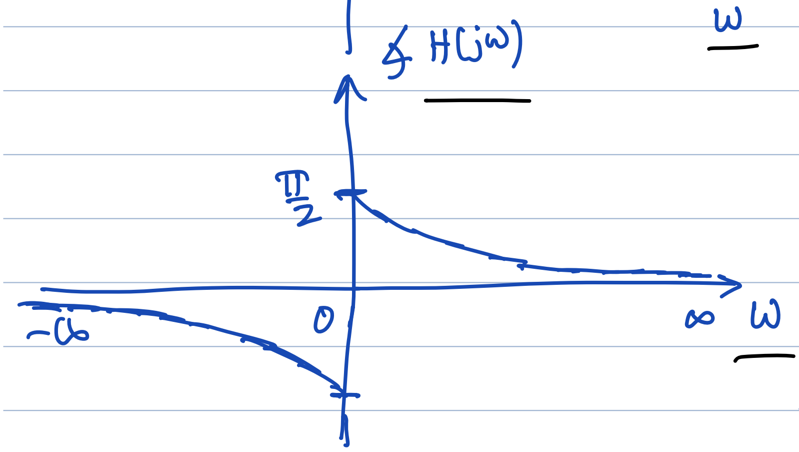
$$\Rightarrow H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{j\omega RC}{1+j\omega RC} = \frac{j\omega RC(1-j\omega RC)}{(\;\cdot\;)}$$

$$|H(j\omega)| = \frac{\omega RC}{\sqrt{1+\omega^2R^2C^2}}, \quad \measuredangle H(j\omega) = \tan^{-1}\left(\frac{\omega RC}{\omega^2R^2C^2}\right)$$

$$= \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

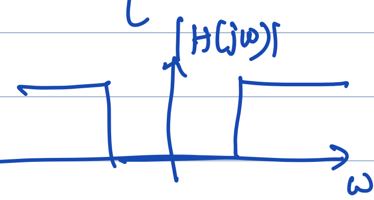
$$\omega \to \infty \quad \longrightarrow 0$$





→ approximates an ideal high pass filter

$$H(j\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

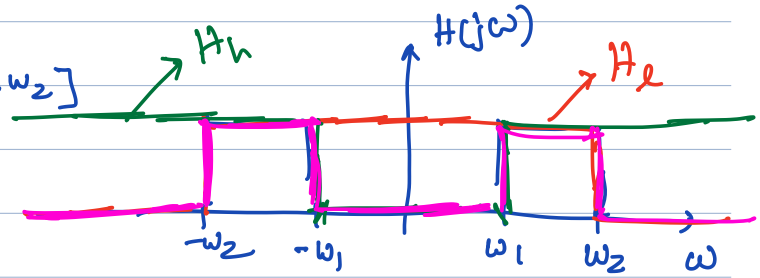


$$\underline{\frac{\pi}{2} - \tan^{-1}(\omega RC)}\ ??$$

(− to be clarified).

Ideal Band Pass Filter

$$H(j\omega) = \begin{cases} 1 & \text{if } |\omega| \in [\omega_1, \omega_2] \\ 0 & \text{otherwise} \end{cases}$$



$\longrightarrow \boxed{H_l(j\omega)} \longrightarrow \boxed{H_h(j\omega)} \longrightarrow$

$\Updownarrow$ when $H(j\omega) = \underbrace{H_l(j\omega)}_{\omega_c = \omega_2} \; \underbrace{H_h(j\omega)}_{\omega_c = \omega_1}$

$\longrightarrow \boxed{H(j\omega)} \longrightarrow$

$\omega_1 < \omega_2$

Suppose we are not allowed to use a high pass filter. How to realize band-pass filter?

since $Y(j\omega) = X(j\omega) H(j\omega)$, $\xrightarrow{X(j\omega)} \boxed{H(j\omega)} \xrightarrow{Y(j\omega)}$

$|Y(j\omega)| = |X(j\omega)| \, |H(j\omega)|$, $\measuredangle Y(j\omega) = \measuredangle X(j\omega) + \measuredangle H(j\omega)$

$\log |Y(j\omega)| = \log |X(j\omega)| + \log |H(j\omega)|$

Bode plot: of $X(j\omega)$ plot of $20 \log_{10} |X(j\omega)|$ vs. ω in log-scale

$\measuredangle X(j\omega)$ vs. $\underline{\omega}$ in log-scale

unit: decibel

$\omega \in (0, \infty)$

Examples

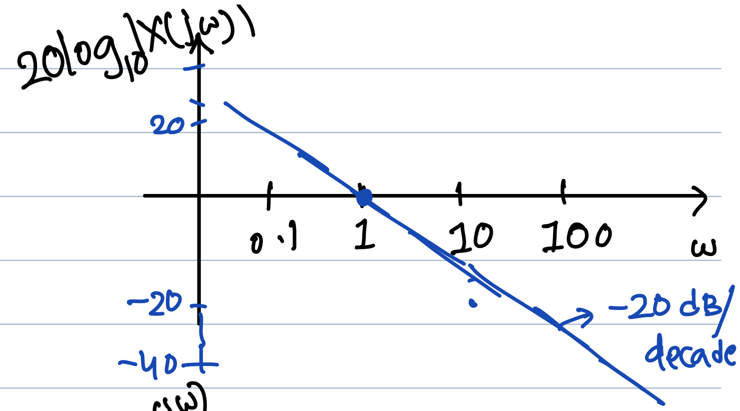
① $X(j\omega) = \dfrac{1}{j\omega}$

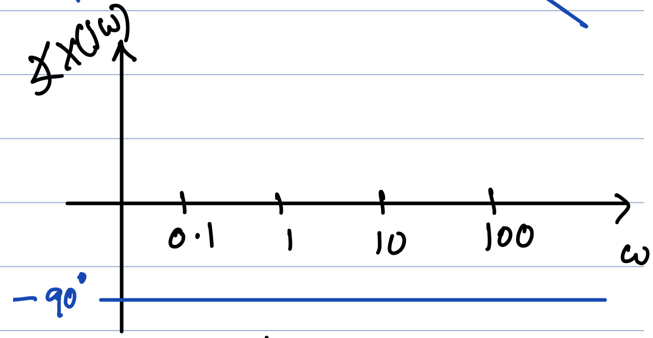
$\measuredangle X(j\omega) = -90^\circ$

$|X(j\omega)| = \dfrac{1}{\omega}$

$20\log_{10}|X(j\omega)|$

$= -20\log_{10}(\omega)$





② $H(j\omega) = \dfrac{1}{1+j\omega\tau}$,

$\tau = RC$

$$|H(j\omega)| \approxeq \begin{cases} 1, & \omega\tau < 0.1 \\ \dfrac{1}{j\omega\tau}, & \omega\tau > 10 \end{cases}$$

$20\log\left(\dfrac{1}{\omega\tau}\right)$

$= -20\log\omega - 20\log\tau$

$= 0$ when $\omega = \dfrac{1}{\tau}$

$$\measuredangle H(j\omega) = \begin{cases} 0^\circ, & \omega = 0 \\ -45^\circ, & \omega = 1/\tau \\ -90^\circ & \omega = \infty \end{cases}$$