$\Rightarrow \frac{4}{5} e^{-j\omega} X(e^{j\omega}) = Y(j\omega) - \frac{4}{5} e^{-j\omega} Y(e^{j\omega})$

$\Rightarrow \frac{4}{5} x[n-1] = y[n] - \frac{4}{5} y[n-1] \Rightarrow y[n] = \frac{4}{5}\left[y[n-1] + x[n-1]\right]$

First Order Discrete-Time System

- A discrete-time first order LTI system is given by:

$$y[n] - ay[n-1] = x[n] \quad \text{with} \quad |a| < 1.$$

- Determine its frequence response $H(e^{j\omega})$ and impulse response $h[n]$.
- Determine its step response $s[n]$.
- The parameter a plays the role of time constant. If $|a|$ is close to 1, the response is slow, and if $|a|$ is close to 0, the response is fast.
- If $a < 0$, we see oscillations and overshoot.

To determine frequency response, apply DTFT on both sides & obtain

$Y(e^{j\omega}) - a e^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega})$

$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a e^{-j\omega}}. \Rightarrow h[n] = a^n u[n]$

If we apply input $s[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$, the output is

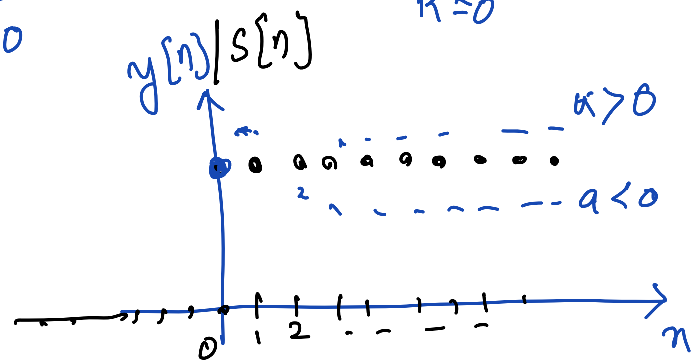
$y[n] = s[n] \circledast h[n]$

$= \sum_{k=-\infty}^{\infty} s[k]\, h[n-k] = \sum_{k=0}^{\infty} a^{n-k} u[n-k] = \sum_{k=0}^{n} a^{n-k}$

$= \frac{1 - a^{n+1}}{1 - a}.$

$y[0] = 1$

$\lim_{n \to \infty} y[n] = \frac{1}{1-a}$



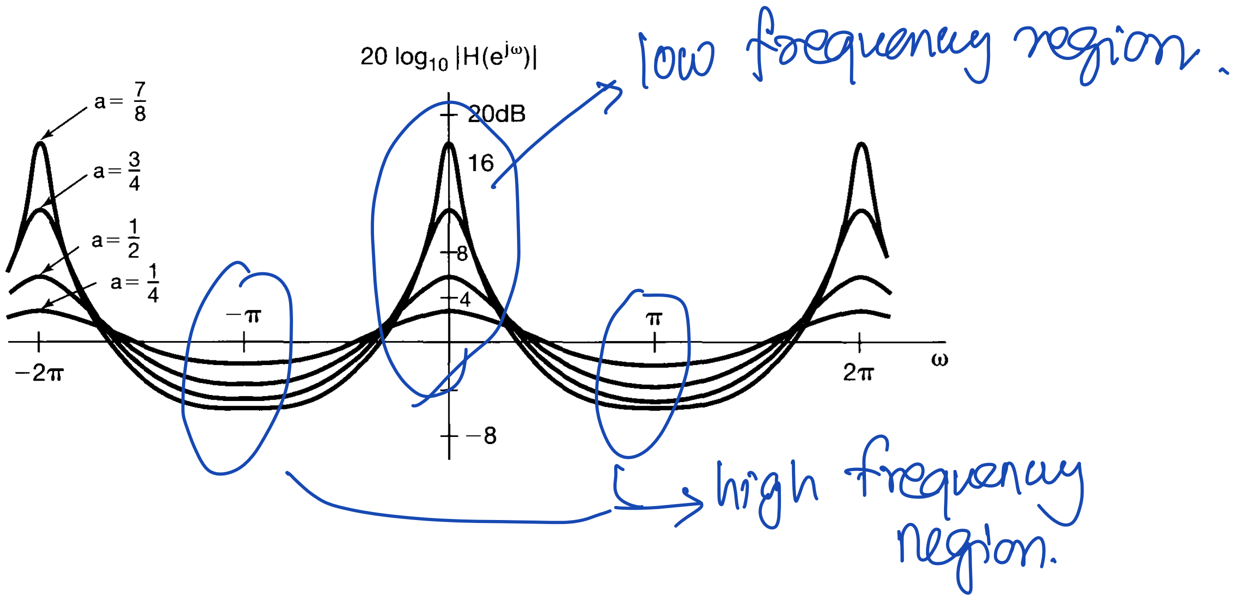
LECTURE 32: 15th October

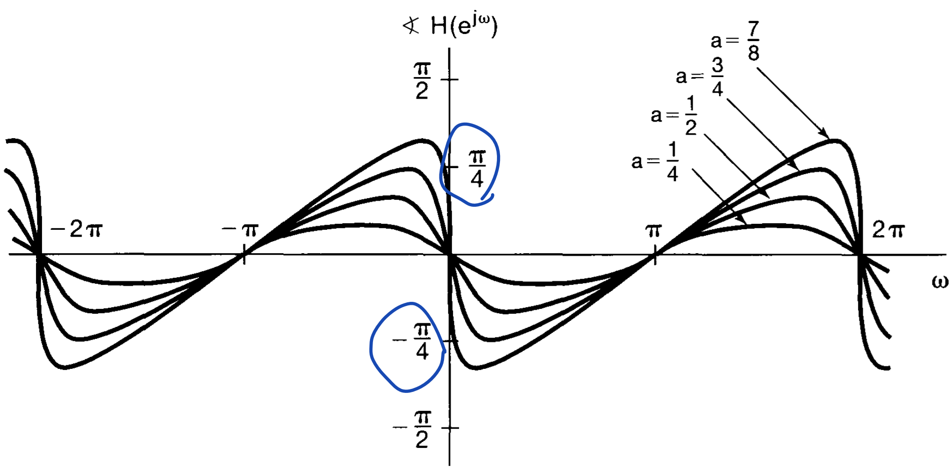
$X(j\omega)$

Log-Magnitude and Phase of Frequency Response

$X(e^{j\omega})$

When $a > 0$:, $a \in (0,1)$

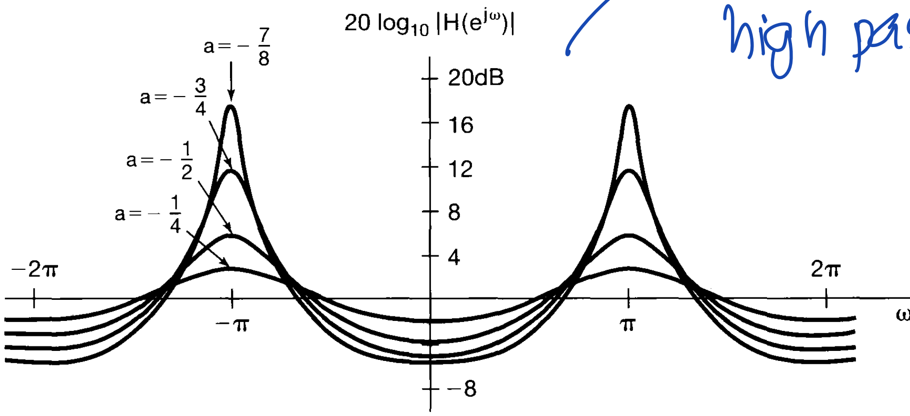


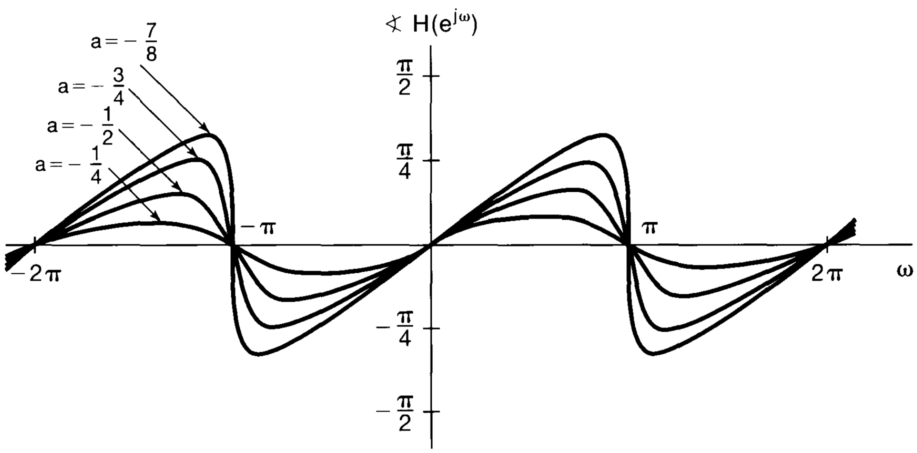


Log-Magnitude and Phase of Frequency Response

When $a < 0$:

approximates a high pass filter





Second Order Discrete-Time System

- A discrete-time ~~first~~ second order LTI system is given by:

$$y[n] - 2r\cos(\theta)y[n-1] + r^2 y[n-2] = x[n],$$

where $r \in (0,1)$ and $0 \le \theta \le 2\pi$.

- Determine its frequence response $H(e^{j\omega})$ and impulse response $h[n]$.

applying DTFT on both sides, we obtain:

$$Y(e^{j\omega}) - 2r\cos\theta\, e^{-j\omega} Y(e^{j\omega}) + r^2 e^{-j2\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta\, e^{-j\omega} + r^2 e^{-j2\omega}}, \quad 2\cos\theta = e^{j\theta} + e^{-j\theta}$$

$$= \frac{1}{\left[1 - re^{j(\theta-\omega)}\right]\left[1 - re^{-j(\theta+\omega)}\right]}$$

$$re^{j\theta}e^{-j\omega}\, re^{-j\theta}e^{-j\omega} = r^2 e^{-j2\omega}$$

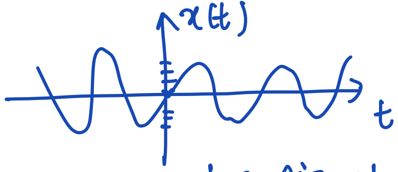
$$-\left[re^{j\theta}e^{-j\omega} + re^{-j\theta}e^{-j\omega}\right] = -2r\cos\theta\, e^{-j\omega}$$

$$= \frac{A}{1 - re^{j\theta}e^{-j\omega}} + \frac{B}{1 - re^{-j\theta}e^{-j\omega}}, \quad A = \frac{1}{1 - e^{-j2\theta}}, \checkmark$$

$$\Rightarrow h[n] = \left[A\left(re^{j\theta}\right)^n + B\left(re^{-j\theta}\right)^n\right]u[n]. \qquad B = \frac{-e^{-j2\theta}}{1 - e^{-j2\theta}}. \checkmark$$

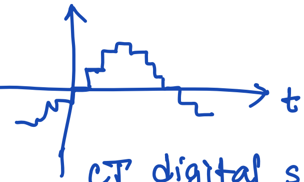
$$= r^n \left[A e^{j\theta n} + B e^{-j\theta n}\right] u[n]$$

$$A\left(1 - re^{-j\theta}e^{-j\omega}\right) + B\left(1 - re^{j\theta}e^{-j\omega}\right) = \frac{1 - \cancel{re^{-j\theta}}e^{-j\omega} - e^{-j2\theta} + \cancel{re^{-j\theta}}e^{-j\omega}}{1 - e^{-j2\theta}} = 1.$$



CT, analog signal

analog to digital conversion



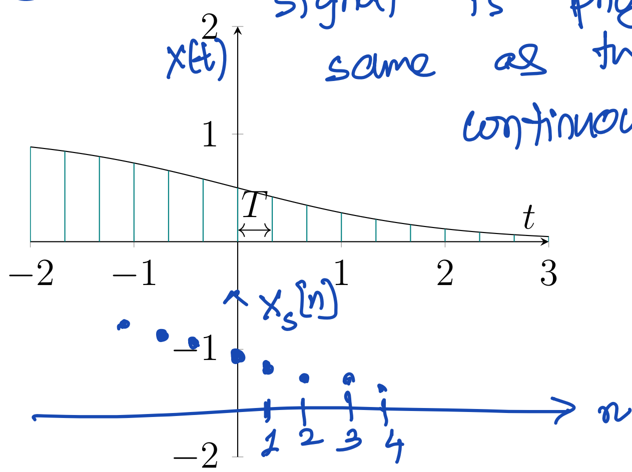
CT, digital signal

: Beyond the scope this course

Sampling

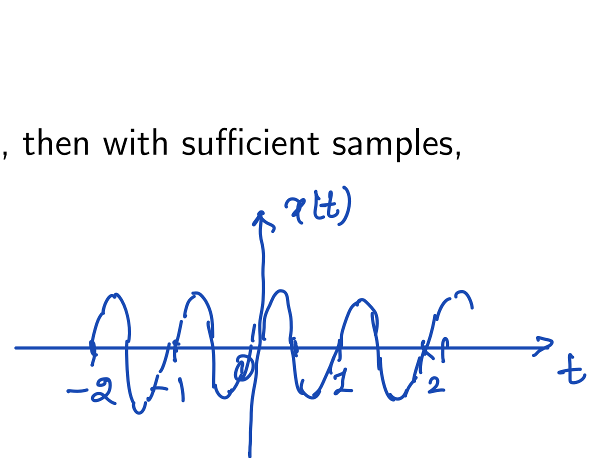
- Question: Can we convert a **continuous-time** signal $x(t)$ to a discrete-time signal?
- One way to do this is to sample at regular intervals of time T and define the sampled signal $x_s[k] := x(kT)$.
- Question: Can we recover a **continuous-time** signal $x(t)$ from regular sampling $x_s[k] = x(kT)$?

the x-axis of the discrete-time signal is physically not the same as the x-axis of the continuous-time signal.



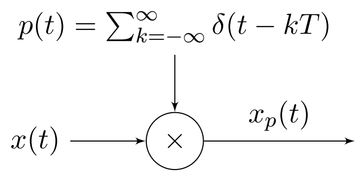
- For example, consider $x(t) = \sin(2\pi t)$.
- Question: Is the recovered signal unique?
- Answer: No in general!
- But if we know the signal $x(t)$ is band-limited, then with sufficient samples, we can!

Let $T = 0.5$, $x_s[k] = 0$, $\forall k \in \mathbb{Z}$



Sampling by Impulse Train

Let us determine $X_p(j\omega)$

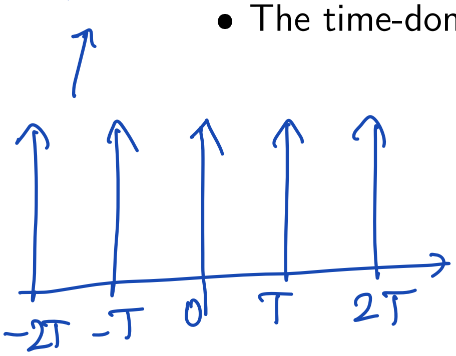


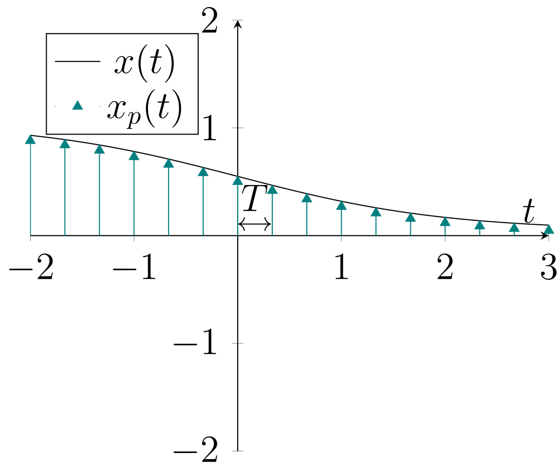
$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \circledast P(j\omega)$$

Figure 1: Diagram of impulse train sampling system.

periodic, with fundamental period T

- The above diagram (system) described *impulse train sampling*.
- The time-domain representation of various signals:





$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_s t}, \quad \omega_s = \frac{2\pi}{T}, \quad a_k = \frac{1}{T} \ \forall k$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{-jk\omega_s t}$$

$$e^{-jk\omega_s t} \longleftrightarrow 2\pi\delta(\omega + k\omega_s)$$

$$\frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2\pi} \delta(\omega - k\omega_s) e^{j\omega t} \, d\omega$$

$$= e^{-jk\omega_s t}$$

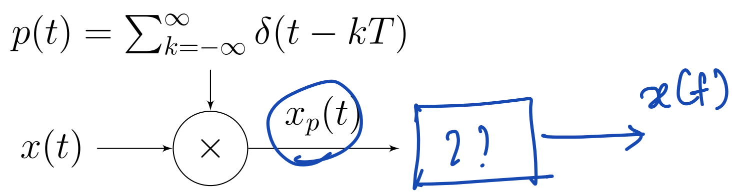
$$P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi \delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi} \left[X(j\omega) \circledast P(j\omega) \right]$$

$$= \frac{1}{\cancel{2\pi}} \frac{\cancel{2\pi}}{T} \sum_{k=-\infty}^{\infty} X(j\omega) \circledast \delta(\omega - k\omega_s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Impulse Train Sampling: Frequency Domain Analysis

$\omega_s = \frac{2\pi}{T}$



- In frequency domain, we have:

$$X_p(j\omega) = \frac{1}{2\pi} P(j\omega) * X(j\omega) \tag{1}$$

- Note that $p(t)$ is periodic with period T.
- Therefore, $p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$, where

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} p(\tau) e^{-jk\omega_s \tau} d\tau = \frac{1}{T}.$$

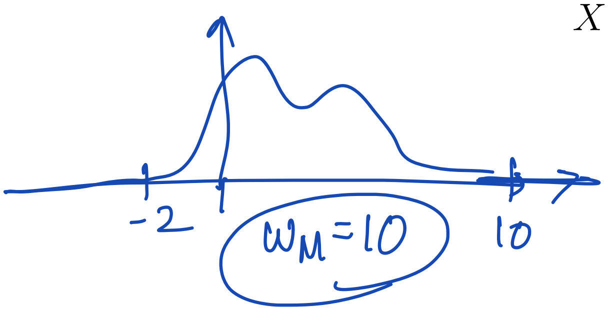
- As a result $P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$.
- Replacing this in (1), we get:

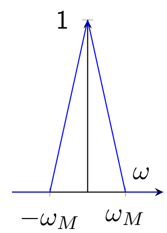
$$X_p(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(j(\omega - k\omega_s)).$$

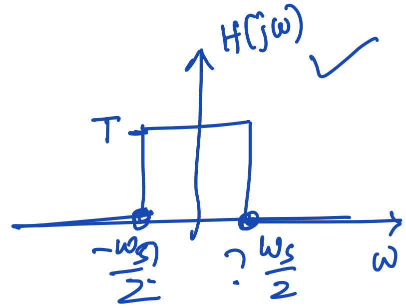
Bandlimited Signals

- Define the bandwidth of a signal $x(t)$ to be the smallest $\omega_M > 0$ such that:

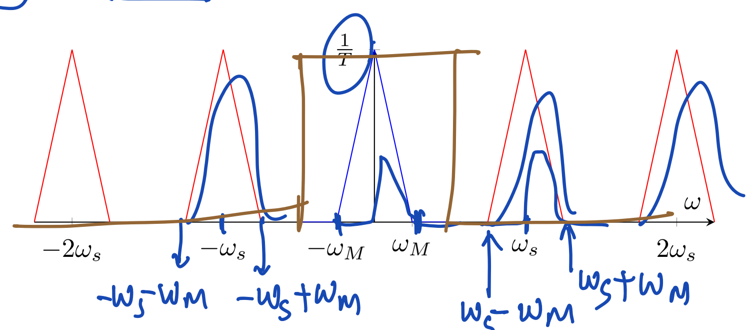
$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M.$$





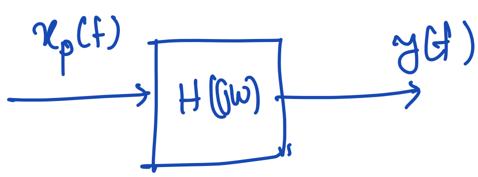


- If $\omega_M < \infty$ we call this signal a *bandlimited signal*.
- So if $\boxed{\omega_s > 2\omega_M}$, for $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ we get:



- What happens when $\omega_s < 2\omega_M$?

For exact reconstruction of $x(t)$ from $x_p(t)$, the following need to be satisfied.



$Y(j\omega) = X_p(j\omega)\,H(j\omega) = X(j\omega)$

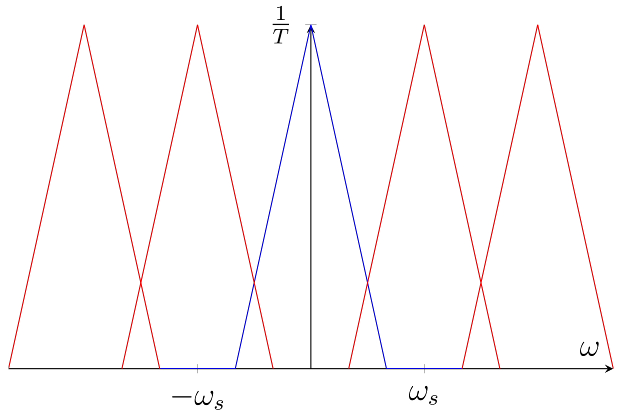
(A) $x(t)$ is bandlimited & its bandwidth ω_M is known.

(B) T is chosen such that $\omega_s = \frac{2\pi}{T} > 2\omega_M \Rightarrow T < \frac{\pi}{\omega_M}$

(C) Availability of ideal low pass filter with cutoff frequency $[\omega_M, \omega_s - \omega_M]$

Aliasing

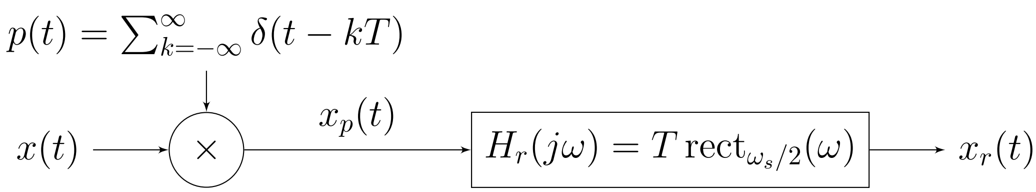
- Overlapping copies of $X(j(\omega - k\omega_0))$ is called *aliasing*.



- In this case, what would be $X_p(j\omega) = \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_0))$?
- Therefore, in order not to have aliasing we need to have: $\omega_s > 2\omega_M$.

Optimal Recovery

- If $x(t)$ is a band-limited signal and $\omega_s > 2\omega_M$, then $x(t)$ can be recovered by (ideal) low-pass filtering of sampled signal $x_p(t)$.
- For perfect recovery, we need to have $H_r(j\omega) = T\,\mathrm{rect}_{\omega_s/2}(\omega)$.

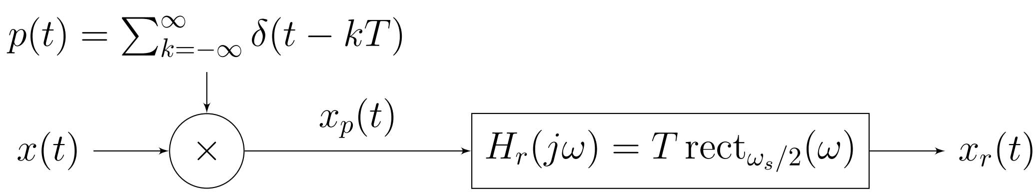


- $2\omega_M$ is called the *Nyquist-rate* (of signal $x(t)$).

Sampling Theorem (Shannon-Nyquist)

Suppose that $x(t)$ is a band-limited signal with bandwidth ω_M. Then $x(t)$ can be recovered from the samples $x(kT)$ for $k = 0, \pm 1, \pm 2, \ldots$ if $\frac{2\pi}{T} = \omega_s > 2\omega_M$. This can be achived by impulse train sampling and recovery system with $H_r(j\omega) = T\,\mathrm{rect}_{\omega_s/2}(\omega)$.

$\underline{T < \frac{\pi}{\omega_M}}$



Questions

1. Let $x(t)$ be a signal band-limited to the interval [-20kHz, 20kHz]. What condition should the sampling period T satisfy to avoid aliasing and allow reconstruction of the signal?

$\omega_M = 2\pi \times 20 \times 1000$

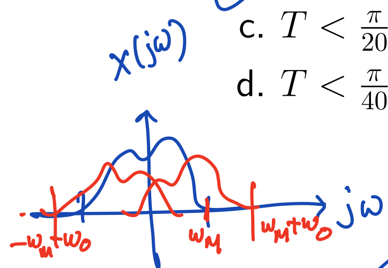
$T < \frac{\pi}{\omega_M} = \frac{1}{40} \times 10^{-3}$

a. $T < \frac{1}{20} \times 10^{-3}$ sec

b. $T < \frac{1}{40} \times 10^{-3}$ sec

c. $T < \frac{\pi}{20} \times 10^{-3}$ sec

d. $T < \frac{\pi}{40} \times 10^{-3}$ sec



$X(j\omega) = 0$ for $|\omega| > \frac{\omega^*}{2}$.

2. Let $x(t)$ be a signal with Nyquist rate given by $\omega^\star$. Determine Nyquist rate of the following signals.

$= 2\omega_M$

a. $x(t) + x(t-1)$ $\longleftrightarrow$ $X(j\omega) + e^{-j\omega} X(j\omega) = (1+e^{-j\omega}) X(j\omega)$ → Nyquist rate ω^*

b. $\frac{d}{dt}x(t)$ $\longleftrightarrow$ $j\omega X(j\omega)$ $\longleftrightarrow$

c. $x(t)\cos(\omega_0 t)$

$= \frac{1}{2} x(t) e^{j\omega_0 t} + \frac{1}{2} x(t) e^{-j\omega_0 t} \longleftrightarrow \frac{1}{2} X(j(\omega-\omega_0)) + \frac{1}{2} X(j(\omega+\omega_0))$

bandwidth is $\frac{\omega^*}{2} + \omega_0$

Nyquist rate : $\omega^* + 2\omega_0$.

3. Suppose signal $x_1(t)$ has bandwidth ω_1 and $x_2(t)$ has bandwidth ω_2. Determine the Nyquist rate of $x_1(t) \times x_2(t)$.

$$x_1(t) \times x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(j\omega) \circledast X_2(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\bar\omega)\, X_2(j(\omega-\bar\omega))\, d\bar\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} X_1(j\bar\omega)\, X_2(j(\omega-\bar\omega))\, d\bar\omega$$

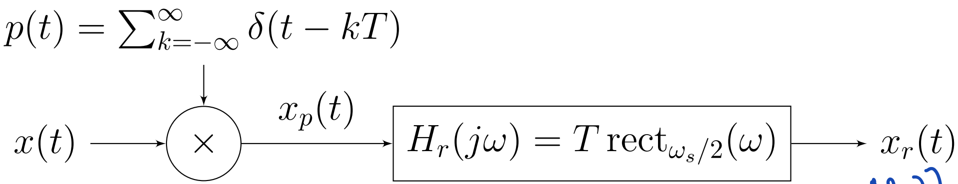
determine the range of ω s.t. $X_2(j(\omega-\bar\omega)) = 0$ when $\bar\omega \in [-\omega_1, \omega_1]$, given that $X_2(j\alpha) = 0$ for $\alpha \in [-\omega_2, \omega_2]$.

$\omega > \omega_1 + \omega_2$ and $\omega < -(\omega_1+\omega_2)$.

Nyquist rate : $2(\omega_1+\omega_2)$.

Reconstruction after Impulse Train Sampling

- A naive approach would be to do linear interpolation.

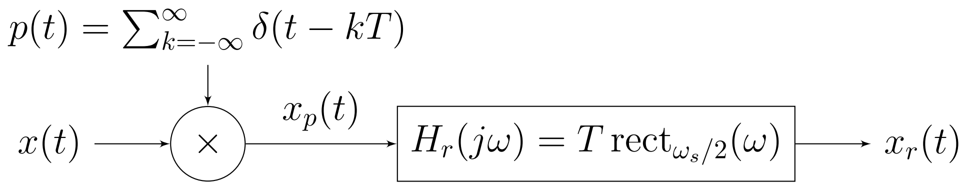


time domain

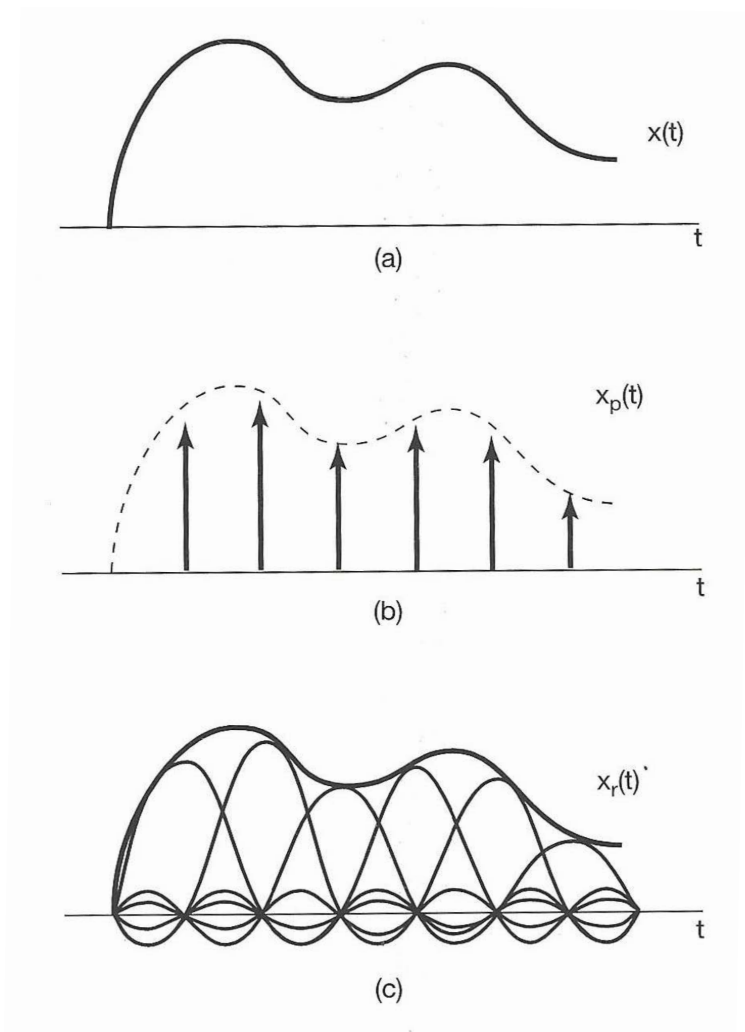
- In TD: $h_r(t) = \frac{T\sin(\frac{\pi}{T}t)}{\pi t}$
- Therefore,

$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right] * h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\left[\delta(t-nT) * h(t)\right] \\
&= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\frac{T\sin(\frac{\pi}{T}(t-nT))}{\pi(t-nT)}
\end{aligned}
$$

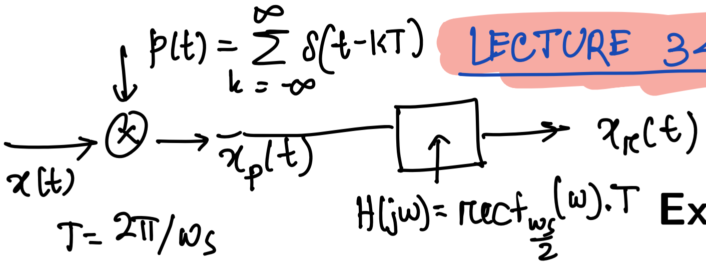
Time Domain View of Reconstruction Process



- $x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\frac{\pi}{T}(t-nT))}{\pi(t-nT)}$
- This is a linear combination of time-shifted sinc functions!



LECTURE 34 & 35: 17th Oct



Exercise

$x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

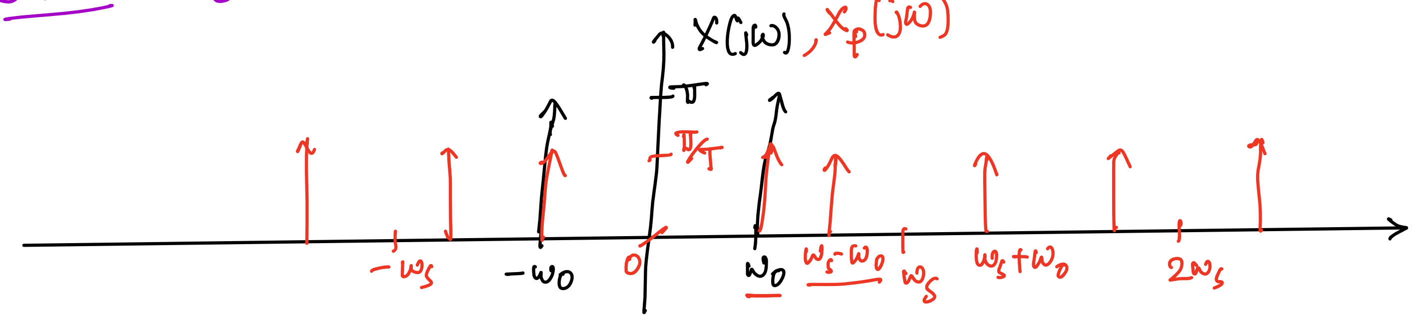
- What is the bandwidth ω_M of ~~cos(t)~~?
- Determine $X_p(j\omega)$, $X_r(j\omega)$, and $x_r(t)$ for $\omega_s = 2\omega_M$, $\omega_s > 2\omega_M$, $\omega_s < 2\omega_M$.

$$X(j\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

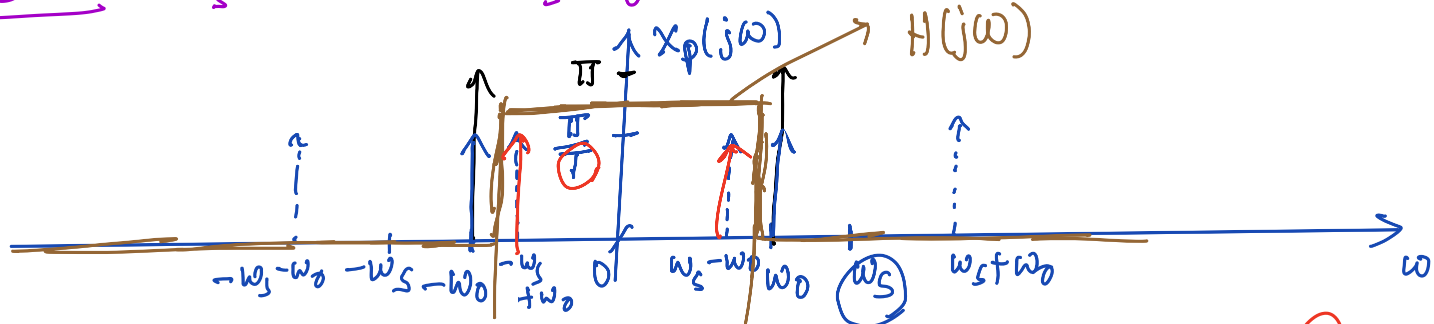
$$X_p(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(j(\omega - K\omega_s))$$

$$H(j\omega) = \begin{cases} 1, & \text{if } |\omega| \le \omega_0 \\ 0 & - - \end{cases}$$

case-1: $\omega_s > 2\omega_0 \iff \omega_s - \omega_0 > \omega_0$



Case 2: $\omega_s < 2\omega_0 \iff \omega_s - \omega_0 < \omega_0$



Suppose, we process $x_p(t)$ with a low pass filter $H(j\omega) = T\,rect_{\frac{\omega_s}{2}}(\omega)$

since $\omega_s < 2\omega_0 \Rightarrow \frac{\omega_s}{2} < \omega_0$

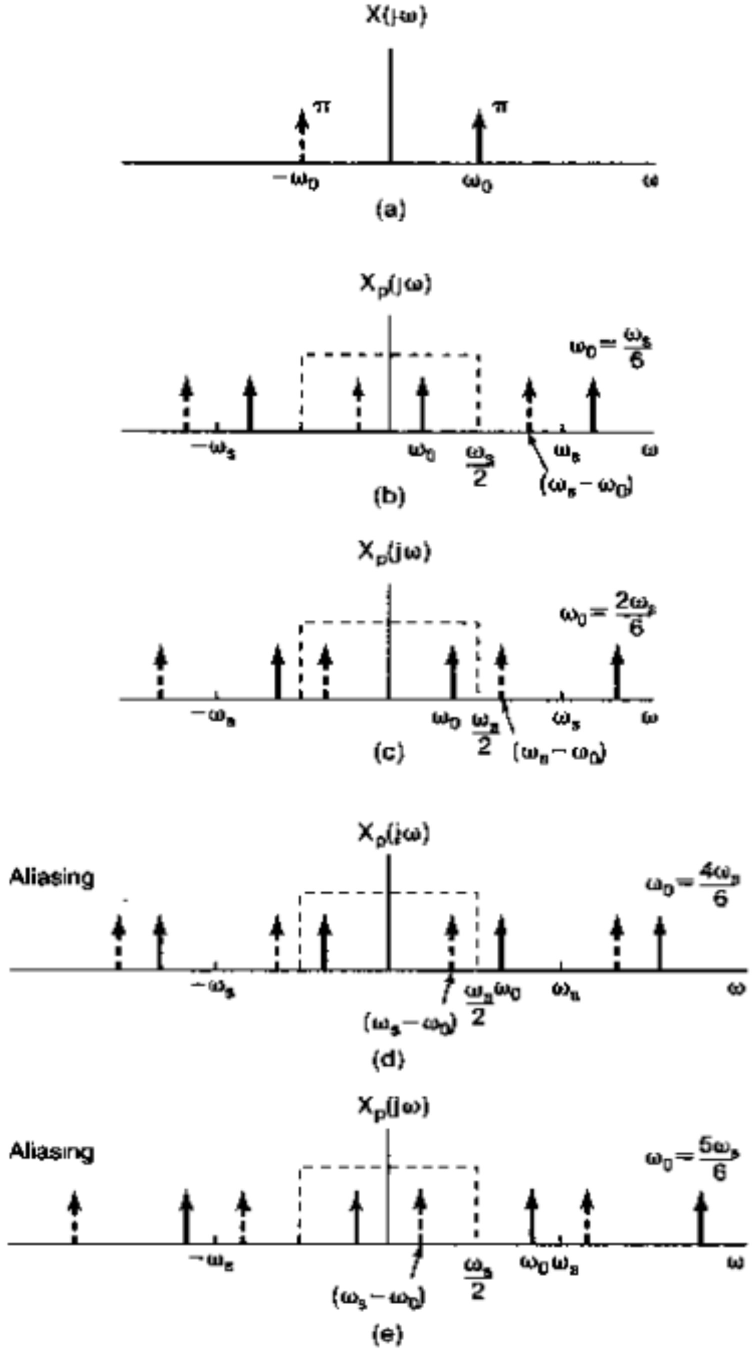
What does $X_r(j\omega)$ look like?

$$X_r(j\omega) = \pi\left[\delta(\omega - \omega_s + \omega_0) + \delta(\omega + \omega_s - \omega_0)\right]$$

$$x_r(t) = \mathcal{F}^{-1}\left[X_r(j\omega)\right] = \cos((\omega_s - \omega_0)t).$$

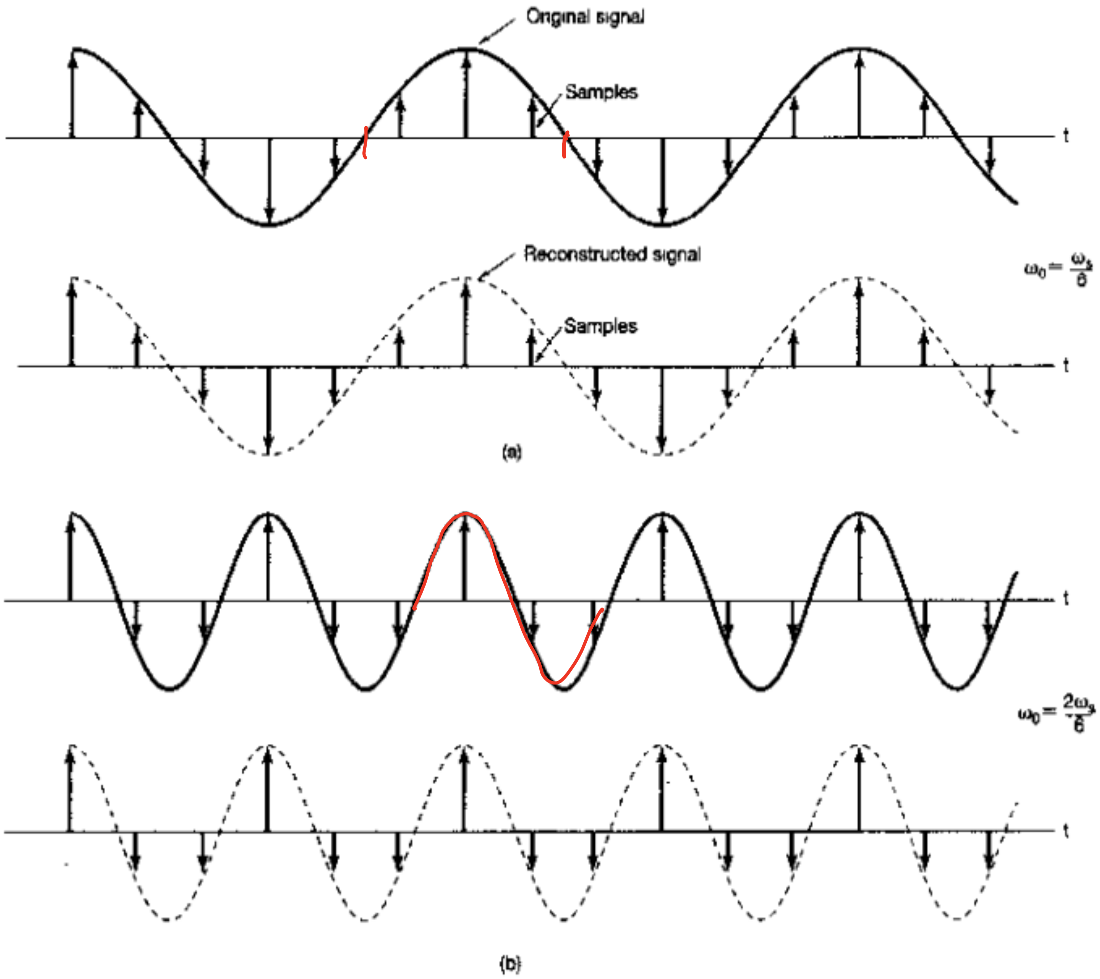
Under-Sampling and Aliasing

- Let $x(t) = \cos(\omega_0 t)$. We vary ω_s by choosing $\omega_s = 6\omega_0, 3\omega_0, 1.5\omega_0, 1.2\omega_0$.



Under-Sampling and Aliasing Continued

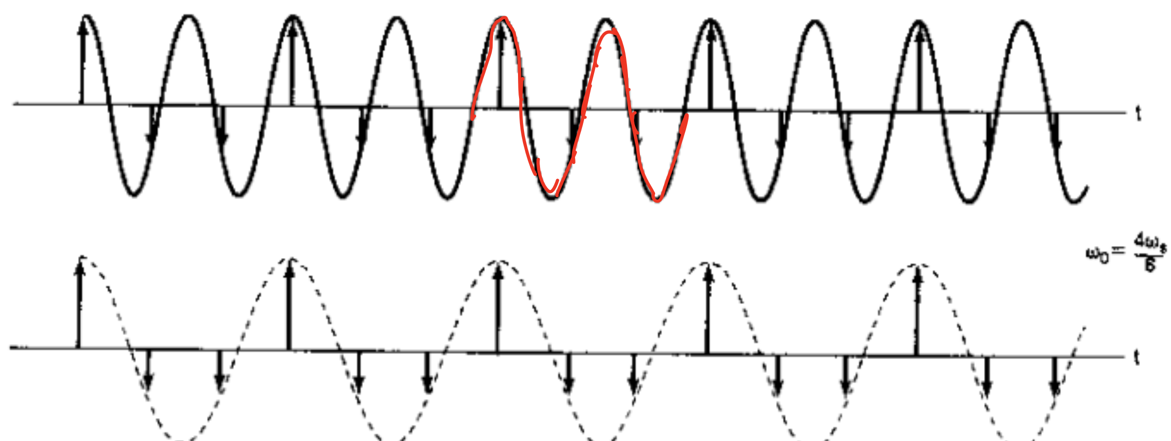
- When $\omega_s = 6\omega_0, 3\omega_0$, the reconstructed signal $x_r(t) = x(t)$.



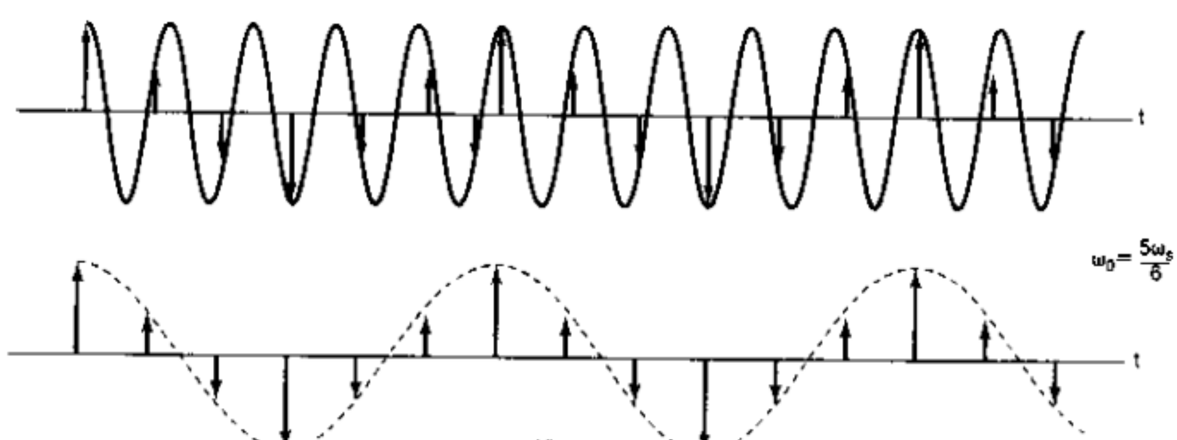
Under-Sampling and Aliasing Continued

$\cos(-\theta) = \cos\theta$

- When $\omega_s = 1.5\omega_0, 1.2\omega_0$, the reconstructed signal $x_r(t) = \cos((\omega_s - \omega_0)t) \neq x(t)$.
- As ω_0 increases relative to ω_s, the frequence of output decreases. Look up **Wagon-Wheel Effect**.
- When $\omega_s = \omega_0$, the output is constant.
- **Phase reversal:** If $x(t) = \cos(\omega_0 t + \phi)$, then $x_r(t) = \cos((\omega_s - \omega_0)t - \phi)$.



(c)



(d)

Problem (7.9) $\quad x(t) = \left(\dfrac{\sin 50\pi t}{\pi t}\right)^2 = y(t) \times y(t)$

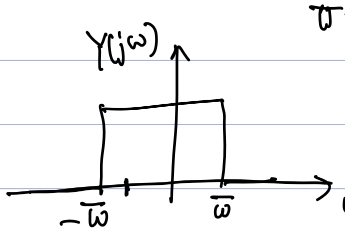
We sample $x(t)$ with $\omega_s = 150\pi$, to obtain $x_p(t)$.

Determine the maximum value of ω_0 s.t. $\quad T = \dfrac{2\pi}{\omega_s} = \dfrac{1}{75}$

$X_p(j\omega) = 75\, X(j\omega)$ for all $|\omega| \leq \omega_0$.

$$X_p(j\omega) = 75 \sum_k X(j(\omega - k\omega_s))$$

Solution $\quad y(t) = \dfrac{\sin 50\pi t}{\pi t} \longleftrightarrow Y(j\omega) = \mathrm{rect}_{50\pi}[\omega]$



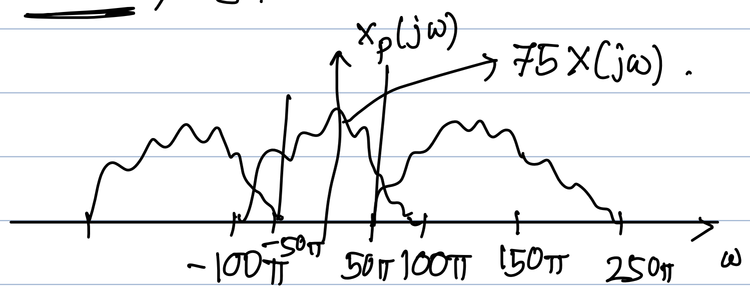
Bandwidth of $y(t) = 50\pi$

$$\mathcal{F}^{-1}\left[\mathrm{rect}_{\bar{\omega}}[\omega]\right] = \frac{1}{2\pi}\int_{-\bar{\omega}}^{\bar{\omega}} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\cdot\frac{1}{jt}\left(e^{j\bar{\omega}t} - e^{-j\bar{\omega}t}\right)$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\cdot 2j\sin\bar{\omega}t$$

$$= \frac{\sin\bar{\omega}t}{\pi t}$$

Bandwidth of $x(t) = 100\pi$, Nyquist rate $= 200\pi$



Thus, $X_p(j\omega) = 75X(j\omega)$

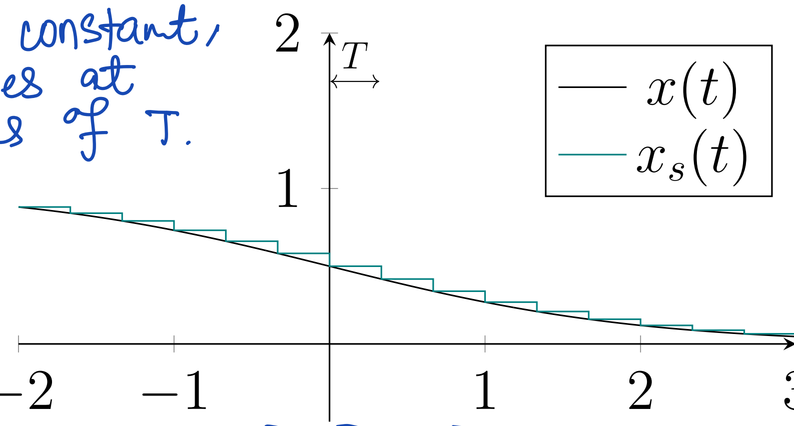
till $|\omega| \leq 50\pi$.

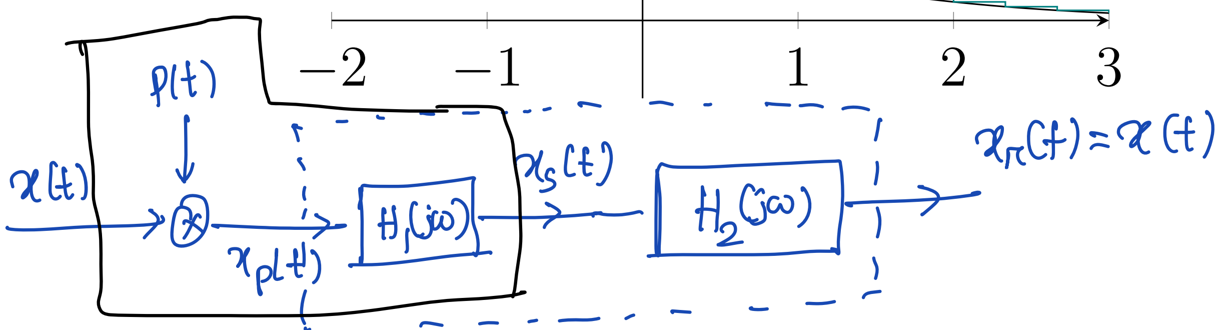
Zero-Order-Hold (ZOH) Sampling

- Since creating an ideal impulse train and implementing an ideal low-pass filter is not possible, in practice, we often do *Zero-Order-Hold* (ZOH) sampling.
- In ZOH sampling, the sampled signal $x_s(t)$ holds the value of $x(t)$ at kth sampling time for T seconds, i.e.,

$$x_s(t) = x(kT) \qquad \text{for } t \in [kT, (k+1)T)$$

piecewise constant, with discontinuities at integer multiples of T.

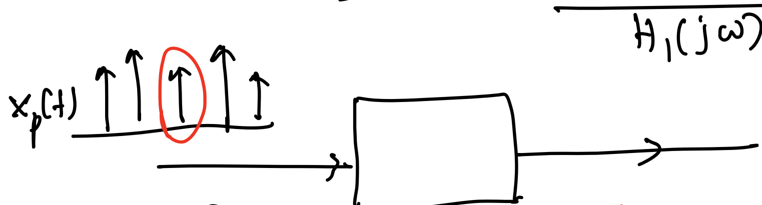


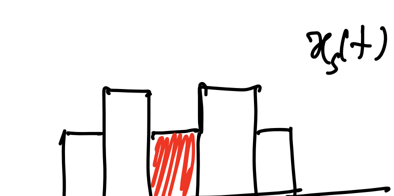


Problem: Given $x_s(t)$, construct $H_2(j\omega)$ s.t. $x_r(t) = x(t)$.

Approach: We will interpret $x_s(t)$ as output of a LTI system when $x_p(t)$ is applied as input to that system.

Then, set $H_2(j\omega) = \dfrac{T\,\mathrm{rect}_{\frac{\omega_s}{2}}(\omega)}{H_1(j\omega)}$.



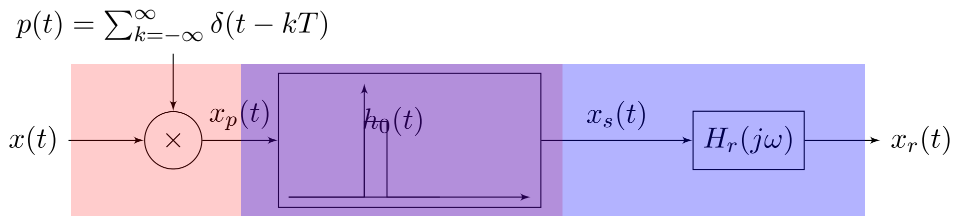


$$h(t) = \begin{cases} 1, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases} = \mathrm{rect}_{\frac{T}{2}}\left(t - \frac{T}{2}\right)$$

$H_1(j\omega) = e^{-j\omega T/2} \frac{2\sin(\omega T/2)}{\omega}$, $H_2(j\omega) = T\left[\text{rect}_{\frac{\omega_s}{2}}(\omega)\right] e^{j\omega T/2} \cdot \frac{\omega}{2\sin(\omega T/2)}$

ZOH Sampling and Perfect Recovery

- ZOH sampling can be viewed as series interconnection of impulse train sampling and an LTI system with the impulse response $h_0(t) = \text{rect}_{T/2}(t - \frac{T}{2})$.



- The main question is that if $\omega_s > 2\omega_M$, then can we *find* a recovery system H_r so that $x_r(t) = x(t)$?
- For perfect recovery, the system in blue box should be the ideal low-pass filter $H_r(j\omega) = T\,\text{rect}_{\omega_s/2}(\omega)$.

ZOH Sampling and Perfect Recovery

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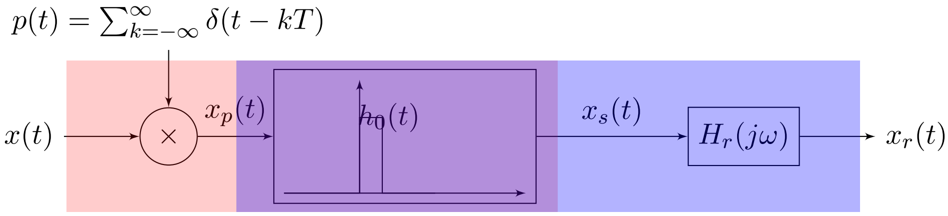
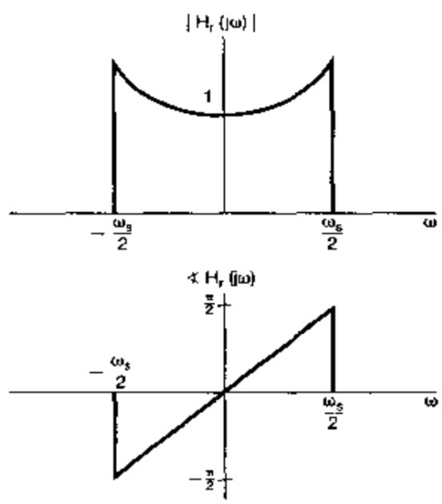


- For perfect recovery, the system in blue box should be the ideal low-pass filter $H_r(j\omega) = T\,\mathrm{rect}_{\omega_s/2}(\omega)$.
- Therefore, for perfect recovery, we need to have:

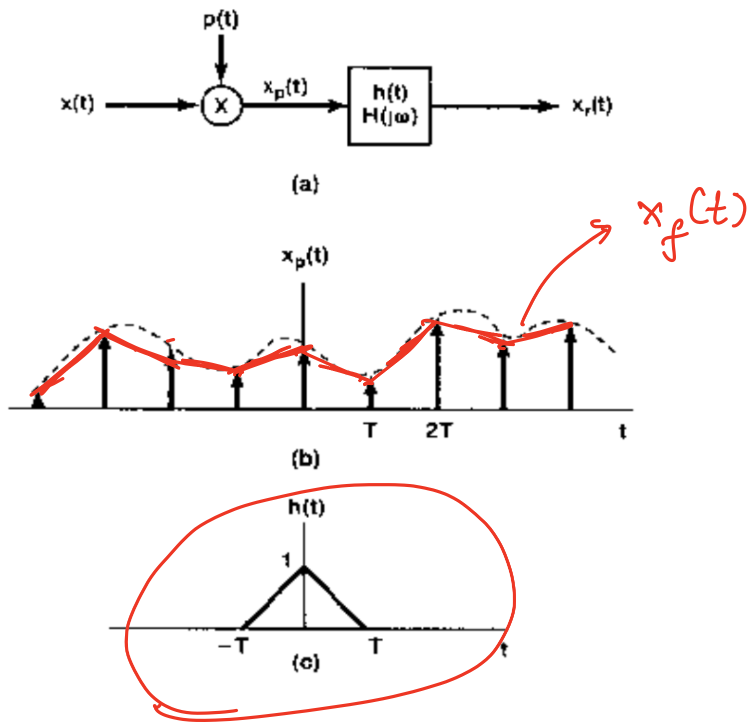
$$H_r(j\omega) = e^{j\omega T/2} \frac{\omega T}{2\sin(T\omega/2)}\,\mathrm{rect}_{\omega_s/2}(\omega).$$

- Note that the first zero of $\frac{\sin(T\omega/2)}{\omega}$ occurs at $\omega_1 = \frac{2\pi}{T} = \omega_s$. Since, $\mathrm{rect}_{\omega_s/2}(\omega_s) = 0$, the zeros of $\frac{\sin(T\omega/2)}{\omega}$ are not going to be problematic in the above fraction.



First Order Hold Sampling

- Corresponds to linear interpolation of the impulse train samples.



DT Processing of CT Signals

$$x_c(t) \longrightarrow \boxed{\text{C\textbackslash D Conversion}} \xrightarrow{x_d[n] = x(nT)} \boxed{\text{DT System } H_d(e^{j\Omega})} \xrightarrow{y_d[n] = y(nT)} \boxed{\text{D\textbackslash C Conversion}} \rightarrow y_c(t)$$

- How to process a CT signal using discrete-time processors?
  1. Convert $x(t)$ to $x_d[n]$ by C/D conversion (with sampling period T)
  2. (design and implement) Filter $x_d[n]$, $X_d(e^{j\Omega})$ in discrete-time with $h_d[n]$, $H_d(e^{j\Omega})$ to get desired $y_d[n]$, $Y_d(e^{j\Omega})$.
  3. Convert $y_d[n]$ to $y(t)$ by D/C conversion (with sampling period T).

$x_p(t) = \sum \alpha \delta(t-\tau)$

$\mathcal{F}(\ ) = \sum \alpha \ \underline{\mathcal{F}[\delta(t-\tau))} = e^{-j\omega\tau}$

Given $x_c(t)$, $x_p(t) = x_c(t) \times p(t) = x(t) \times \left(\sum_{K=-\infty}^{\infty} \delta(t-KT)\right)$

$$X_p(j\omega) = \sum_{K=-\infty}^{\infty} x_c(KT)\, \mathcal{F}\left[\delta(t-KT)\right] \leftrightarrow e^{-j\omega KT}$$

$$= \sum_{K=-\infty}^{\infty} x_c(KT)\,\delta(t-KT)$$

DT Processing of CT Signals

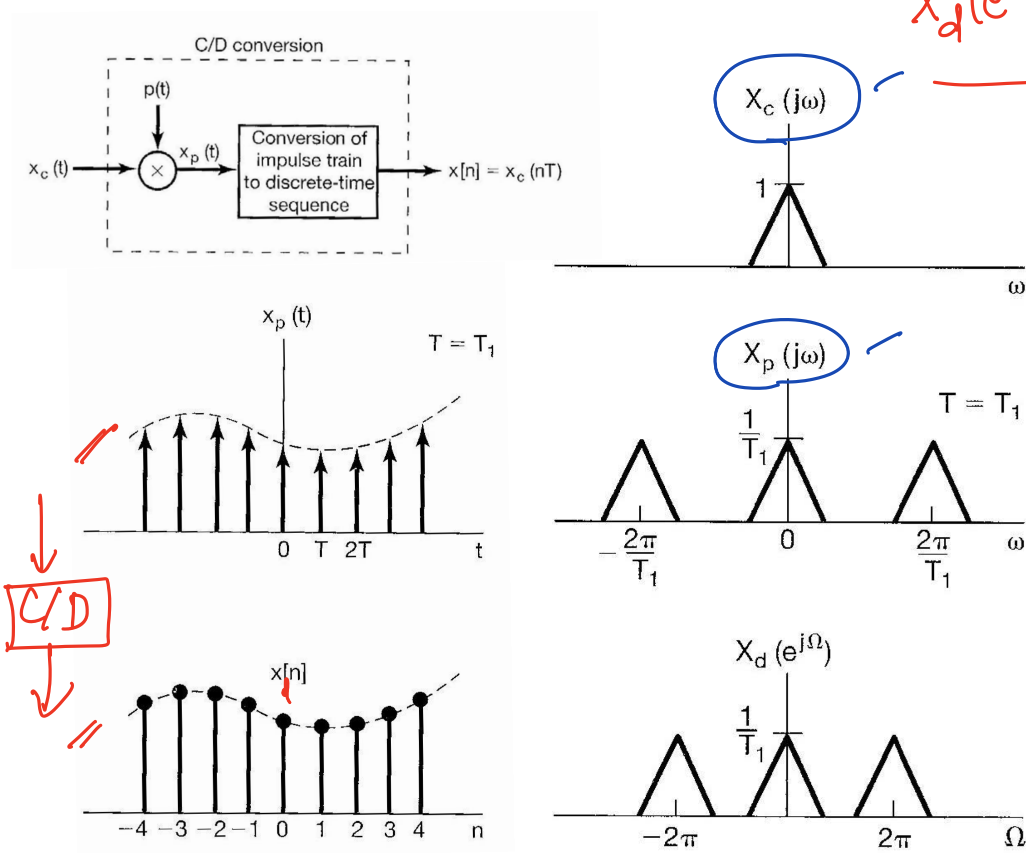
impulse train sample.

$$x_c(t) \longrightarrow \boxed{\text{C\D Conversion}} \xrightarrow{x_d[n]=x(nT)} \boxed{\text{DT System } H_d(e^{j\Omega})} \xrightarrow{y_d[n]=y(nT)} \boxed{\text{D\C Conversion}} \longrightarrow y_c(t)$$

- The discrete-time sequence $x_d[n] = x_c(nT)$.
- What is the relationship between $X_c(j\omega)$ and $X_d(e^{j\Omega})$?
- We use ω to denote frequency in continuous-time FT and Ω to denote frequency in discrete-time FT.

$$\sum_{K=-\infty}^{\infty} x_d[K]\, e^{-j\Omega K} = \sum_{K=-\infty}^{\infty} x_c(KT)\, e^{-j\Omega K}$$

$$X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right)$$



DT Processing of CT Signals

- CT Sampling - Perspective 1: $x_p(t) = x_c(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$ and

$$\boxed{X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))}$$

- CT Sampling - Perspective 2: On the other hand $\delta(t - nT) \longleftrightarrow e^{-j\omega nT}$ and hence:

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

- DT Sampling: $x_d[n] = x(nT)$ for n and hence

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega n}$$

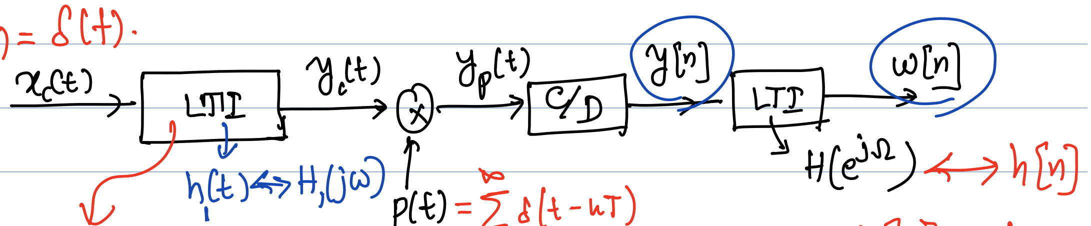
- Conclusion:

$$X_d(e^{j\Omega}) = X_p(j\Omega/T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega/T - k\omega_s))$$

- Converting $x_p(t)$ to $x_d[n]$ scales the time-axis by $1/T$. Consequently, in the frequency domain, $X_d(e^{j\Omega})$ is obtained by scaling $X_c(j\omega)$ by factor T, i.e., $\Omega = T\omega$.

Problem 7.30

Suppose $x_c(t) = \delta(t)$.



$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t)$$

Determine $h[n]$ s.t. $w[n] = \delta[n]$.

$$W_n(e^{j\Omega}) = 1$$

Solution:

$$H_1(j\omega) = \frac{1}{1+j\omega}$$

$$Y_c(j\omega) = \frac{1}{1+j\omega}$$

$$Y_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left(j\left(\omega - k\frac{2\pi}{T}\right)\right)$$

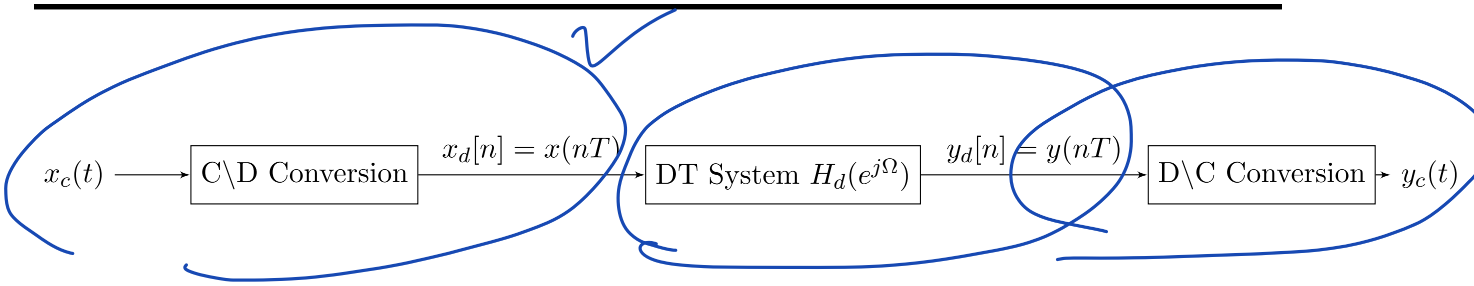
$$Y_n(e^{j\Omega}) = Y_p\left(j\frac{\Omega}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left(j\left(\frac{\Omega-2\pi k}{T}\right)\right)$$

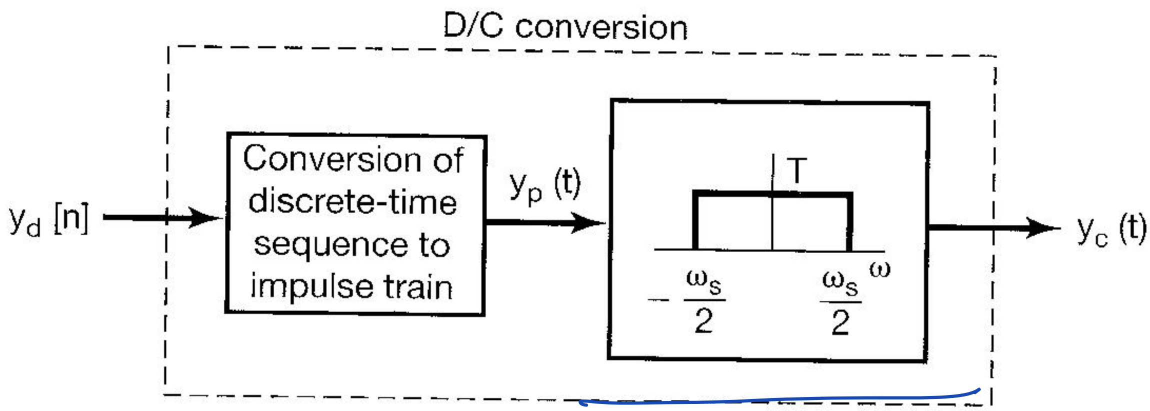
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\left[\frac{1}{1+j\left(\frac{\Omega-2\pi k}{T}\right)}\right]$$

$$H(e^{j\Omega}) = \left(\qquad\right)^{-1}.$$

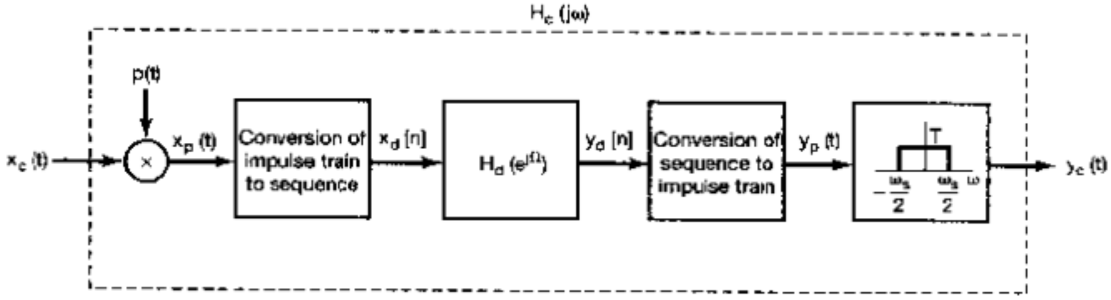
Find $h[n]$: Homework.

DT Processing of CT Signals





DT Processing of CT Signals



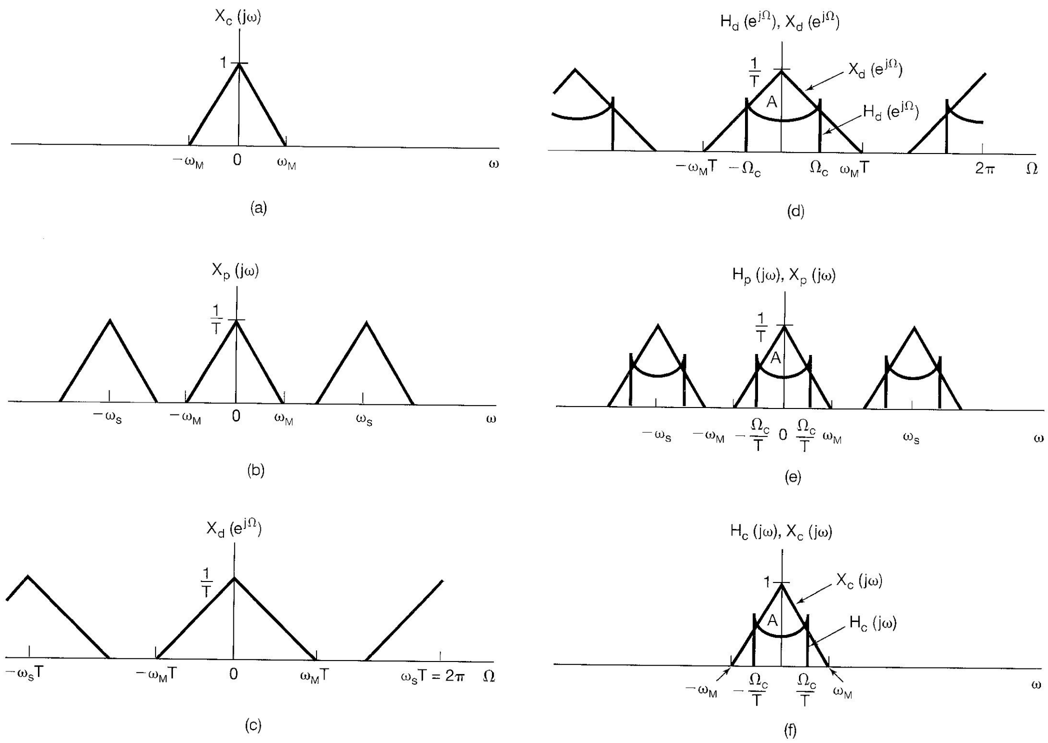


Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.