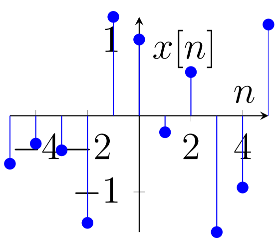
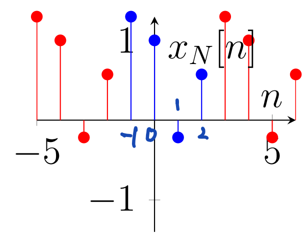
LECTURE 29 : 9th October

Discrete-Time Fourier Transform

- Let $x[n]$ be an arbitrary DT signal.





- For an even $N \geq 1$, define $x_N[n]$ to be the signal that:
  a. is periodic with period N ($\omega_0 = \frac{2\pi}{N}$),
  b. $x_N[n] = x[n]$ for $n = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$.
- The FS representation of $x_N[n]$ would be:

$$x_N[n] = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} a_k e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}, \tag{1}$$

where $a_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} x[n] e^{-jk\omega_0 n}$.

define
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- For large N, we have:

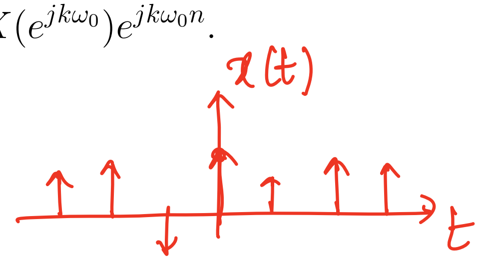
$$a_k = \frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} x[n] e^{-jk\omega_0 n} \approx \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n} =: \frac{1}{N} X(e^{jk\omega_0}), \tag{2}$$

where $X(e^{j\omega_0})$ is the Fourier Transform, if $x[n]$ is viewed as a continuous-time signal $x(t)$ with impulses at integer values of t.

- Replacing (2) in (1), we get:

$$x_N[n] \approx \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \omega_0 X(e^{jk\omega_0}) e^{jk\omega_0 n}.$$

Suppose we have a continuous time signal:
$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \delta(t-k)$$



$X(j\omega) = \sum_{k=-\infty}^{\infty} \alpha_k e^{-jk\omega}$

Continued.

- Replacing (??) in (??), we get:

2 1

$$x_N[n] \approx \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \omega_0 X(e^{jk\omega_0}) e^{jk\omega_0 n}.$$

- Since the fundamental period $\omega_0 = \frac{2\pi}{N}$ approaches 0, $k\omega_0$ approaches the continumm

when $k = \frac{N}{2}$, $k\omega_0 = \frac{N}{2} \cdot \frac{2\pi}{N}$
$= \pi$

- Note that $\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \omega_0 X(e^{jk\omega_0}) e^{jk\omega_0 n} \approx \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$.

$(\omega = k\omega_0)$

- Letting N go to infinity, we get:

$$x[n] = \lim_{N\to\infty} x_N[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

Fourier Transform of DT Signals

For a discrete-time signal $x[n]$, we have:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{(Synthesis Equation)}$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(Analysis Equation).}$$

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn(\omega+2\pi)}$$
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \underbrace{e^{-jn2\pi}}_{=1}$$
$$= X(e^{j\omega}).$$

- Notation: $x[n] \longleftrightarrow X(e^{j\omega})$
- $X(e^{j\omega})$ is finite for all ω if either (a) $x[n]$ is absolutely summable or (b) $x[n]$ has finite energy.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- DTFT differs from CTFT in the following two ways.
  1. $X(e^{j\omega})$ is periodic in ω with period 2π. Thus, it suffices to specify it over any contiguous interval of ω of length 2π. For a continuous-time signal $x(t)$, its FT $X(j\omega)$ is not necessarily periodic.
  2. The inverse DTFT integral is computed over an interval of length 2π even though $X(e^{j\omega})$ is defined for $\omega \in (-\infty, \infty)$.
  3. Notation: $x[n] \longleftrightarrow X(e^{j\omega})$ while $x(t) \longleftrightarrow X(j\omega)$.

Recall that $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

DTFT Example

- What is the DTFT of $x_1[n] = \delta[n]$? $X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$

- What is the DTFT of $x_2[n] = \delta[n] + \delta[n-1] + \delta[n+1]$?

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \delta[n+1]\right) e^{-j\omega n}$$

$$= e^{-j\omega \cdot 0} + e^{-j\omega \cdot 1} + e^{+j\omega 1}$$

$$= 1 + e^{j\omega} + e^{-j\omega} = 1 + 2\cos(\omega).$$

– periodic w/ fn period 2π.

$$X_3(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

$|ae^{-j\omega}| = |a| < 1$

- What is the DTFT of $x_3[n] = a^n u[n]$ for $|a| < 1$?
- We have:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

- $|X(e^{j\omega})|^2 =$ $X(e^{j\omega}) \cdot X(e^{j\omega})^*$

$$= \frac{1}{1 - ae^{-j\omega}} \cdot \frac{1}{1 - ae^{j\omega}}$$

$$= \frac{1}{1 + a^2 - ae^{j\omega} - ae^{-j\omega}} = \frac{1}{1 + a^2 - 2a\cos\omega}$$

DTFT of $a^n u[n]$

Note that: $\frac{1}{1-2a\cos\omega+a^2}$ is maximized (minimized) when $-2a\cos\omega$ is minimized (maximized)

Case A. If $0 < a < 1$, then $\left|X(e^{j\omega})\right|^2$ achieves maximum when $\omega = 0$, and $\left|X(e^{j\omega})\right|^2$ achieves minimum when $\omega = \pi$. Thus,

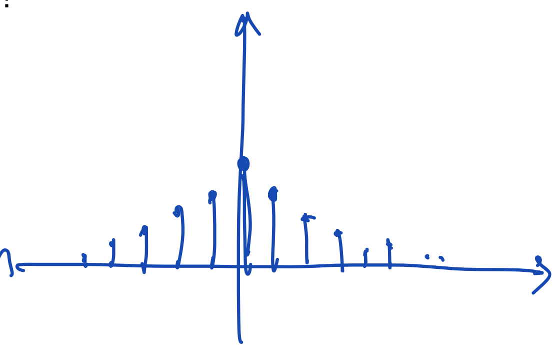
$$\max\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1-2a+a^2} = \frac{1}{(1-a)^2}$$
$$\min\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1+2a+a^2} = \frac{1}{(1+a)^2}.$$

Case B: If $-1 < a < 0$, then $\left|X(e^{j\omega})\right|^2$ achieves maximum when $\omega = \pi$, and $\left|X(e^{j\omega})\right|^2$ achieves minimum when $\omega = 0$. Thus,

$$\max\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1+2a+a^2} = \frac{1}{(1+a)^2}$$
$$\min\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1-2a+a^2} = \frac{1}{(1-a)^2}.$$

DTFT of $a^{|n|}$

- What is the DTFT of $x[n] = a^{|n|}$ for $|a| < 1$?



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} a^{|n|} e^{-j\omega n} + \sum_{n=1}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} a^{-n} e^{-j\omega n} + ae^{-j\omega} \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{m=0}^{\infty} (ae^{j\omega})^{m} + ae^{-j\omega} \sum_{n=0}^{\infty} (ae^{-j\omega})^{n}$$

$$= \frac{1}{1-ae^{j\omega}} + \frac{ae^{-j\omega}}{1-ae^{-j\omega}} = \frac{1-ae^{-j\omega}+ae^{-j\omega}-a^2}{1+a^2-2a\cos\omega}$$

$$= \frac{1-a^2}{1+a^2-2a\cos\omega},$$ periodic with period 2π.

Exercise

- Note that $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$. Now, given the discrete-time signal $x[n]$ with Fourier transform $X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{j\omega}}$, calculate $x[n]$.

  A. $x[n] = 2^n u[-n]$

  B. $x[n] = 2^{-n} u[-n]$

  C. $x[n] = 2^{n+2} u[n+2]$

$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-ae^{j\omega}}$.

$\left(\frac{1}{2}\right)^{-n} u[-n] = 2^n u[-n]$.

If $x[n] \longleftrightarrow X(e^{j\omega})$, then, $x[-n] \longleftrightarrow X(e^{-j\omega})$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$x[-n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{-j\omega n} \, d\omega = \frac{1}{2\pi} \int_{2\pi} X(e^{j\bar{\omega}}) e^{j\bar{\omega} n} \, d\bar{\omega},$$

$\bar{\omega} = -\omega$

DTFT of Periodic Signals

- Let $x[n]$ be a periodic signal with period N. What is the FT of this signal?
- By the Fourier series decomposition of periodic signals, we have:

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}.$$

$$a_k e^{jk\frac{2\pi}{N}n} \updownarrow 2\pi a_k \sum_{l=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N} - 2\pi l\right)$$

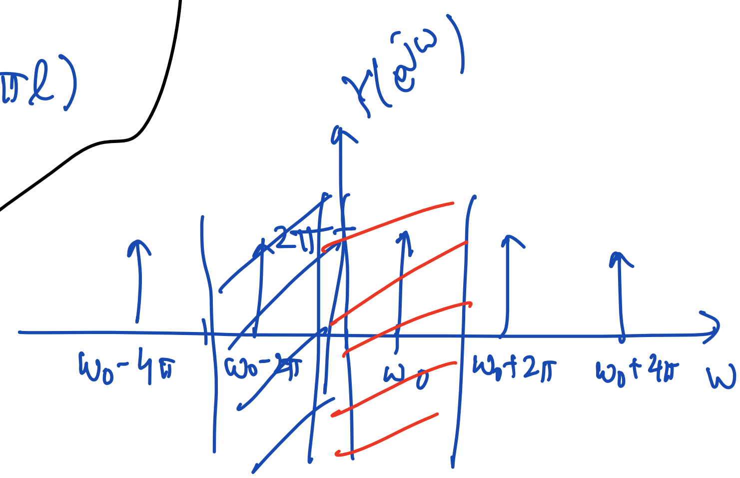
- Show that $e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$ ✓
- Using this and linearity of Fourier Transform, we get:

$$\cancel{X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right).}$$

Let $Y(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$

$$y[n] = \frac{1}{2\pi} \int_{2\pi} Y(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{\cancel{2\pi}} e^{j\omega_0 n} \cdot \cancel{2\pi}$$



$$X(e^{j\omega}) = \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N} - 2\pi l\right)$$

period : N

$\omega = \frac{2\pi}{N}$.

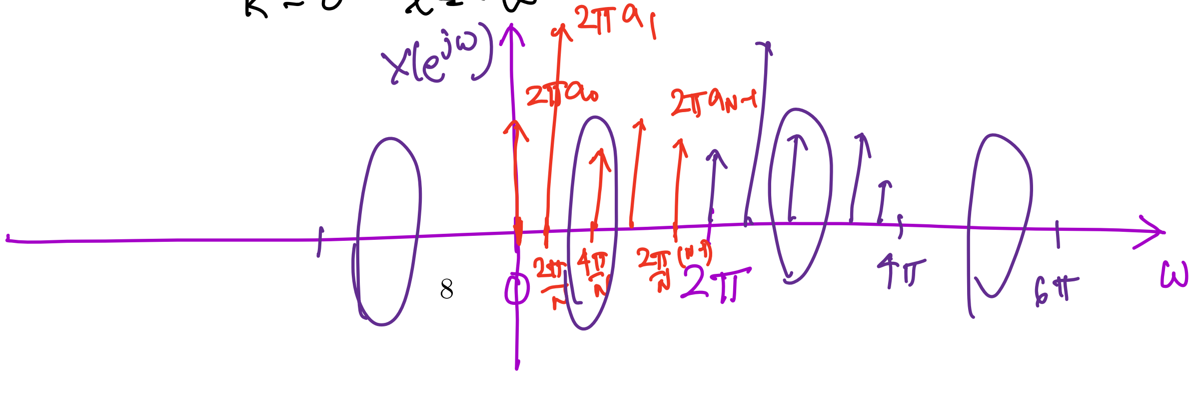
$a_0 \longleftrightarrow 0$

$a_1 \longleftrightarrow \frac{2\pi}{N}$

$a_2 \longleftrightarrow \frac{4\pi}{N}$

$\vdots$

$a_{N-1} \longleftrightarrow \frac{2\pi}{N}(N-1)$

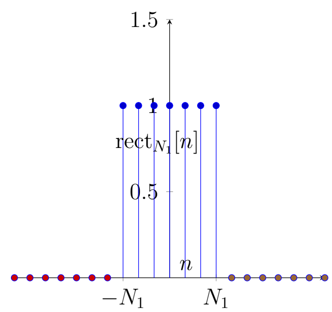


LECTURE 30 & 31 : 10th October

DTFT of Rectangular Pulse

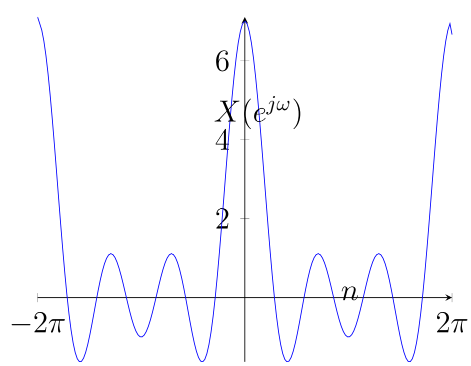
- FT of discrete-time rect

$$x[n] = \mathrm{rect}_{N_1}[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & \text{otherwise} \end{cases}$$



- Recall that $1 + \beta + \beta^2 + \ldots + \beta^{r-1} = \frac{1-\beta^r}{1-\beta}$.
- For $\omega \neq 0$, we have:

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = e^{-j\omega N_1} \frac{e^{2j\omega N_1 + 1} - 1}{e^{j\omega} - 1}$$



Properties of DTFT

- Periodicity of DTFT: $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$
- Linearity: $\alpha x[n] + \beta y[n] \longleftrightarrow \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
- Time-Reversal: $x[-n] \longleftrightarrow X(e^{-j\omega})$
- Time shift: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

$x[n] \longleftrightarrow X(e^{j\omega})$

$$\sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j\omega(n-n_0)} \underbrace{e^{-j\omega n_0}}$$

$$= e^{-j\omega n_0} \left[\sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j\omega(n-n_0)} \right] = e^{-j\omega n_0} X(e^{j\omega})$$

- Frequency shift: $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)})$$

- Conjugation: $x^*[n] \longleftrightarrow (X(e^{-j\omega}))^*$

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n} \Rightarrow [X(e^{-j\omega})]^* = \sum_{n=-\infty}^{\infty} x[n]^* (e^{j\omega n})^*$$

$$= \sum_{n=-\infty}^{\infty} x[n]^* e^{-j\omega n}$$

- Show that if $x[n]$ is real and even, then $X(e^{j\omega})$ is also a real and even function of ω.

If $x[n]$ is real $\Rightarrow x[n] = x[n]^* \Rightarrow X(e^{j\omega}) = (X(e^{-j\omega}))^*$

If $x[n]$ is even $\Rightarrow x[n] = x[-n] \Rightarrow X(e^{j\omega}) = X(e^{-j\omega})$

If $x[n]$ is both real and even, (1) $X(e^{-j\omega}) = X(e^{-j\omega})^*$

$\Rightarrow X(e^{j\omega})$ is real.

(2) $X(e^{j\omega}) = X(e^{-j\omega})$

$\Rightarrow X(e^{j\omega})$ is even.

Properties of DTFT

- DT Differentiation: $x[n] - x[n-1] \longleftrightarrow (1 - e^{-e^{j\omega}})X(e^{j\omega})$

$x[n] \longleftrightarrow X(e^{j\omega})$

$x[n-1] \longleftrightarrow e^{-j\omega} X(e^{j\omega})$

- DT integration (accumulation):

$$\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2k\pi).$$

- Corollary: Fourier transform of the unit step function $u[n]$ is given by

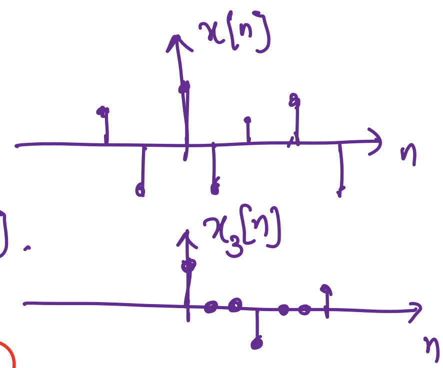
$$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2k\pi).$$

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right], & \text{when } n \text{ is a multiple } k \\ 0, & \text{otherwise.} \end{cases}$$

If $x[n] \longleftrightarrow X(e^{j\omega})$, find DTFT of $x_{(k)}[n]$.

$\updownarrow$

$X(e^{jk\omega})$



$$\sum_{n=-\infty}^{\infty} x_{(k)}[n]\, e^{j\omega n} = \sum_{m=-\infty}^{\infty} x[m]\, e^{j\omega k m} = X(e^{j\omega k})$$

$m = l \longleftrightarrow n = lk$

Properties of DTFT cont.

- Time-scaling: Define

$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}] & \text{for } n \text{ a multiple of } k \\ 0 & \text{otherwise} \end{cases}.$$

Then,

$$x_{(k)}[n] \longleftrightarrow X(e^{j\omega k}).$$

- Differentiation in frequency domain:

$$nx[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega}).$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\,(-jn)\,e^{-j\omega n}$$

$$j\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} n\,x[n]\,\cancel{(j(-j))}^{\,1}\, e^{-j\omega n}$$

- Parseval's Relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

holds only when $x[n]$ has finite energy / square integrable.

For periodic DT signals, use Parseval's relation given for Fourier Series coefficients.

DTFT Properties

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals	$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

Exercise

Determine the DTFT of the signal

$$x[n] = \begin{cases} 1, & n \in \{0, 2, 4\} \\ 2, & n \in \{1, 3, 5\} \\ 0, & \text{otherwise.} \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = 1 + e^{-j2\omega} + e^{-j4\omega}$$

$$+ 2e^{-j\omega} + 2e^{-j3\omega} + 2e^{-j5\omega}$$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[-m]e^{j\omega m}$

$\hookrightarrow$ mth FS of $X(e^{j\omega})$ is $x[-m]$.

Duality of DTFT and Continuous-time Fourier Series

- We noticed that $X(e^{j\omega})$ is always periodic with period 2π.
- Let $z(t) = X(e^{jt})$ which is periodic with period 2π.
- T,hen $z(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt}$ where:

$$a_k = \frac{1}{2\pi}\int_{2\pi} z(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{-jk\omega}d\omega = x[-k].$$

- $x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) \xleftrightarrow{FS.} x[-n]$.

For a continuous-time periodic signal, FS coefficients span from $-\infty$ to ∞.

For a discrete-time signal that spans from $-\infty$ to ∞, its DTFT is a continuous function of ω which is periodic with period 2π.

Properties of DTFT: Convolution Property

- As in the case of CT, synthesis equation provides *decomposition of a discrete-time signal into exponential components.*

- From convolution, we have

$$y[n] = x[n] * h[n]$$

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \xrightarrow{\quad H(e^{j\omega})e^{j\omega n} \quad}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega n} e^{-j\omega k} = e^{j\omega n}\left[\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}\right] = H(e^{j\omega})e^{j\omega n}$$

- Therefore:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega$$

- Therefore: For any signals $x[n]$ and $h[n]$, we have:

$$y[n] = x[n] * h[n] \longleftrightarrow X(e^{j\omega})\underbrace{H(e^{j\omega})}_{} = Y(e^{j\omega}).$$

DTFT of impulse response, also called transfer fn. of the system.

- Multiplication:

$$y[n] = x[n] \times z[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Z(e^{j\omega - j\theta})d\theta.$$

Exercise

Consider an LTI System with inpulse response $h[n] = \alpha^n u[n]$ with $|\alpha| < 1$. Let the input to this system be $x[n] = \beta^n u[n]$ with $|\beta| < 1$. Determine $y[n]$ when $\alpha = \beta$ and $\alpha \neq \beta$.

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}, \quad H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}.$$

Case 1: $\alpha \neq \beta$

$$Y(e^{j\omega}) = \frac{A}{1-\beta e^{-j\omega}} + \frac{B}{1-\alpha e^{-j\omega}},$$

$$A(1-\alpha e^{-j\omega}) + B(1-\beta e^{-j\omega}) = 1$$

$$\Rightarrow A = \frac{-\beta}{\alpha-\beta}$$

$$B = \frac{\alpha}{\alpha-\beta}$$

$$y[n] = A\beta^n u[n] + B\alpha^n u[n]$$

Case 2: $\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

$$j\frac{d}{d\omega}H(e^{j\omega}) = j\frac{-1}{(1-\alpha e^{-j\omega})^2}\cdot -\alpha(-j)e^{-j\omega}$$

$$= j\frac{-j\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$= \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$n\alpha^n u[n] \longleftrightarrow \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$(n+1)\alpha^{n+1}u[n+1] \longleftrightarrow \cancel{e^{j\omega}}\cdot\left[\frac{\alpha\ \cancel{e^{-j\omega}}}{(1-\alpha e^{-j\omega})^2}\right]$$

$$\Rightarrow \underbrace{(n+1)\alpha^n u[n+1]}_{y[n]=} \longleftrightarrow \frac{1}{(1-\alpha e^{-j\omega})^2}$$

$$= (n+1)\alpha^n u[n]$$

Exercise

Solve **Example 5.14** from the book.

Frequency Response of Systems Characterized by Constant-Coefficient Difference Equation

- Similar to CT, in DT, many practical systems are defined by **constant coefficient difference equations**

$$
\begin{aligned}
&a_N y[n-N] + \cdots + a_1 y[n-1] + a_0 y[n] \\
&\qquad = b_M x[n-M] + \cdots + b_1 x[n-1] + b_0 x[n].
\end{aligned}
$$

- Taking FT of both sides we get:

$$
\begin{aligned}
&(a_N e^{-j\omega N} + \cdots + a_1 e^{-j\omega} + a_0) Y(e^{j\omega}) \\
&\qquad = (b_M e^{-j\omega M} + \cdots + b_1 e^{-j\omega} + b_0) X(e^{j\omega})
\end{aligned}
$$

- Therefore,

$$
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_M e^{-j\omega M} + \cdots + b_1 e^{-j\omega} + b_0}{a_N e^{-j\omega N} + \cdots + a_1 e^{-j\omega} + a_0}.
$$

Problems

- Problem 1: What is the frequency response of a system whose input-output behavior is given by:

$$y[n] - y[n-1] = 3x[n] + 2x[n-1] + x[n-2].$$

applying DTFT on both sides, we obtain

$$Y(e^{j\omega}) - e^{-j\omega} Y(e^{-j\omega}) = 3X(e^{-j\omega}) + 2e^{-j\omega} X(e^{-j\omega}) + e^{-j2\omega} X(e^{-j\omega})$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{3 + 2e^{-j\omega} + e^{-j2\omega}}{1 - e^{-j\omega}}.$$

- Problem 2: Suppose $x[n] \longleftrightarrow X(e^{j\omega})$. Determine DTFT of the following signals in terms of $X(e^{j\omega})$.
  - $x_1[n] = x[-1-n] + x[1-n]. \rightarrow 2\cos\omega \, X(e^{-j\omega})$
  - $x_2[n] = (n-1)^2 x[n].$

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

$$x[-1-n] \longleftrightarrow e^{j\omega} X(e^{-j\omega})$$

$$= x[-(n+1)]$$

$$x[1-n] = x[-(n-1)] \longleftrightarrow e^{-j\omega} X(e^{-j\omega})$$

$$x_2[n] = n^2 x[n] - 2n x[n] + x[n]$$

$$X_2(e^{j\omega}) = -\frac{d^2}{d\omega^2} X(e^{j\omega}) - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

- Problem 3: When $x[n] = (\frac{4}{5})^n u[n]$ is applied to a DT LTI system, the output $y[n] = n(\frac{4}{5})^n u[n]$. Determine the difference equation relating $x[n]$ and $y[n]$.

$$X(e^{j\omega}) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}, \quad Y(e^{j\omega}) = j\frac{d}{d\omega} X(e^{j\omega}) = j\,\frac{-1 \cdot -\frac{4}{5}(-j)}{(1 - \frac{4}{5}e^{-j\omega})^2}.$$

$$= \frac{4}{5} \cdot \frac{e^{-j\omega}}{(1 - \frac{4}{5}e^{-j\omega})^2}$$

$$H(e^{j\omega}) = \frac{\frac{4}{5} e^{-j\omega}}{(1 - \frac{4}{5} e^{-j\omega})} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$\Rightarrow \frac{4}{5} e^{-j\omega} X(e^{j\omega}) = Y(j\omega) - \frac{4}{5} e^{-j\omega} Y(e^{j\omega})$

$\Rightarrow \frac{4}{5} x[n-1] = y[n] - \frac{4}{5} y[n-1] \Rightarrow \underline{y[n] = \frac{4}{5}\left[y[n-1] + x[n-1]\right]}$

First Order Discrete-Time System

- A discrete-time first order LTI system is given by:

$$y[n] - ay[n-1] = x[n] \quad \text{with} \quad |a| < 1.$$

- Determine its frequence response $H(e^{j\omega})$ and impulse response $h[n]$.
- Determine its step response $s[n]$.
- The parameter a plays the role of time constant. If $|a|$ is close to 1, the response is slow, and if $|a|$ is close to 0, the response is fast.
- If $a < 0$, we see oscillations and overshoot.