EE61012 – Spring Semester 2024-25

Homework 2: Convex Optimization in Control and Signal Processing

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Q 2.1: Alternative Form of Farka's Lemma

Following the above theorem, show that exactly one of the following two sets is empty.

$$S_2 = \{x \in \mathbb{R}^n | Ax \leq b\},$$
$$T_2 = \{y \in \mathbb{R}^m | y \geq 0, y^\top A = 0, y^\top b > 0\}.$$

Show that S_2 can be written as S_1, perhaps by using some ideas discussed in class while looking at equivalence of optimization problems. Then, construct the alternative system for the new S_1 system and show that the alternative system is nothing but T_2. Hint: any x can be written as $x_+ - x_-$ with both $x_+ \geq, x_- \geq 0$.

Q 2.2: LP Complementarity Slackness

Consider the following primal and dual pair of linear optimization problems.

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & Ax = b, x \geq 0, \end{aligned}$$

and

$$\begin{aligned} \text{minimize}_{y \in \mathbb{R}^m} \quad & -b^\top y \\ \text{subject to} \quad & A^\top y \leq c. \end{aligned}$$

Show that if x^* and y^* are the respective optimal solutions, then

$$(x^*)^\top [A^\top y^* - c] = 0.$$

In other words, whenever $x_i^* \neq 0$, the i-th constraint in $A^\top y \leq c$ holds with equality. This property is known as the *complementaty slackness condition.*

Q 2.3: LP Duality

Consider the following linear optimization problem:

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^4} \quad & x_1 + 5x_2 + 2x_3 + 13x_4 \\ \text{subject to} \quad & 5x_1 - 6x_2 + 4x_3 - 2x_4 = 0, \\ & x_1 - x_2 + 6x_3 + 9x_4 = 16, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Find the dual of the above optimization problem. Suppose $x_1 = 0, x_2 = 2, x_3 = 3$, and $x_4 = 0$ is the optimal solution of the above problem. Following complementarity slackness condition, find the optimal dual solution and show that strong duality holds. Verify your answer using CVX/YALMIP.

Q 2.4: LP Duality

Consider the following linear optimization problem:

$$\begin{aligned} \text{maximize}_{x\in\mathbb{R}^2} \quad & 5x_1 + 10x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \le 50, \\ & 4x_1 + 2x_2 \le 60, \\ & x_1 \le 5 \\ & x_1, x_2 \ge 0. \end{aligned}$$

Find the dual of the above optimization problem. Suppose $x_1 = 5, x_2 = 15$ is the optimal solution of the above problem. Following complementarity slackness condition, find the optimal dual solution and show that strong duality holds. Verify your answer using CVX/YALMIP.

Q 2.5: Convex Optimality Conditions

Consider the convex optimization problem:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i \in [m]. \end{aligned}$$

Let there exist x^* and λ^* that satisfy the KKT conditions for this problem. Then, show that

$$\nabla f(x^*)^\top (x - x^*) \ge 0,$$

for all feasible x. This shows the equivalence between the general optimality condition derived in class and the KKT conditions derived in the context of duality.

Q 2.6: Strongly Convex Functions

Show that if a function is strongly convex, it has a unique minimizer.

Q 2.7: LP Duality, midsem 2023-24

Determine dual of the following optimization problem and simplify it as much as possible. Determine the optimal solution and the optimal value of the dual problem.

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^3} \quad & x_1 + x_3 \\ \text{subject to} \quad & x_1 + 2x_2 \le 5, \quad x_1 + 2x_3 = 6, \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

Q 2.8: Convex duality, midsem 2023-24

Determine the dual of the following optimization problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}} \quad & x^2+1 \\ \text{subject to} \quad & (x-2)(x-4) \leq 0, \end{aligned}$$

and simplify it as much as possible. Determine the optimal solution and the optimal value of both the primal and the dual problems.

Q 2.9: Optimality conditions, midsem 2023-24

Verify that $x^\star = (1, 0.5, -1)$ is an optimal solution of the following problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^3} \quad & \frac{1}{2}x^\top Px + q^\top x + r \\ \text{subject to} \quad & -1 \leq x_i \leq 1, \quad \text{for} \quad i=1,2,3, \\ \text{where} \quad & P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 2 \end{bmatrix}, \quad q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \quad r = 1. \end{aligned}$$

Q 2.10: Convex optimization, midsem 2023-24

Determine if the following problem is a convex optimization problem.

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^2} \quad & |x_1+5| + |x_2-3| \\ \text{subject to} \quad & 2.5 \leq x_1 \leq 5, \quad -1 \leq x_2 \leq 5. \end{aligned}$$

Q 2.11: Convex duality, endsem 2023-24

Determine dual of the following optimization problem and simplify it as much as possible.

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^n, t\in\mathbb{R}} \quad & \frac{1}{2}x^\top x + ct \\ \text{subject to} \quad & Ax = b + te, t \geq 0, \end{aligned}$$

where e is the vector of suitable dimension with all entries 1, and A, b, c are known quantities.

Q 2.12: Linear Support Vector Classification with Soft Margin

Given labeled dataset $\{x^i, y^i\}_{i\in[N]}$ where each $x^i \in \mathbf{R}^n$ is associated with a label $y^i \in \{1, -1\}$ such that $y^i = 1$ if $x^i \in A$ and $y^i = -1$ if $x^i \in B$. Consider the following classification problem with soft margin:

$$\begin{aligned} \min_{w\in\mathbb{R}^n, b\in\mathbb{R}, \epsilon\in\mathbb{R}^N} \quad & \frac{1}{2}||w||_2^2 + C\sum_{i=1}^{N} \epsilon_i \\ \text{s.t.} \quad & 1 - y^i(w^\top x^i + b) \leq \epsilon_i, \qquad \forall i \in [N], \\ & \epsilon_i \geq 0, \qquad \forall i \in [N]. \end{aligned}$$

Show that the dual of the above problem is given by:

$$\begin{aligned} \min_{\lambda \in \mathbb{R}^N} \quad & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j y^i y^i (x^i)^\top x^j + \sum_{i=1}^{N} \lambda_i \\ \text{s.t.} \quad & 0 \leq \lambda_i \leq C, \quad \forall i \in [N], \\ & \sum_{i=1}^{N} \lambda_i y^i = 0. \end{aligned}$$

Given the optimal dual solution λ^*, find the optimal primal solution w^* and b^*.

Q 2.13: Equivalent formulation of SVM

Show that the Support Vector Classification problem can be stated equivalently as:

$$\min_{w \in \mathbb{R}^n, b \in \mathbb{R}} \quad \lambda ||w||_2^2 + \frac{1}{N}\sum_{i=1}^{N} \max\left(0, 1 - y^i(w^\top x^i + b)\right).$$

How does λ relate with C in the previous formulation?

Q 2.14: Support vector regression

Given dataset $\{x^i, y^i\}_{i \in [N]}$ where each $x^i \in \mathbf{R}^n$ and $y^i \in \mathbb{R}$ and linear hypothesis $f(x) = w^\top x + b$, the ϵ-support vector regression problem is given by

$$\begin{aligned} \min_{w \in \mathbb{R}^n, b \in \mathbb{R}} \quad & \frac{1}{2}||w||_2^2 \\ \text{s.t.} \quad & y^i \leq w^\top x^i + b + \epsilon, \qquad \forall i \in [N], \\ & y^i \geq w^\top x^i + b - \epsilon, \qquad \forall i \in [N]. \end{aligned}$$

Find the dual of the above problem and show how to derive the primal optimal solution from the dual optimal solution.

Q 2.15: Midsem Spring 2022-23

Consider the following primal optimization problem:

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^2} \quad & -x_2 \\ \text{subject to} \quad & x_2 \geq 0, x_1 \geq 0, \\ & x_1 - x_2 \leq 3. \end{aligned}$$

Find the dual of the above optimization problem. Determine whether the primal is infeasible, unbounded or has an optimal solution. Determine the dual optimization problem and whether the dual is infeasible, unbounded or has an optimal solution. If either one has an optimal solution, then show that strong duality holds.

Q 2.16: Midsem Spring 2022-23

Let $S_2 = \{x \in \mathbb{R}^n | x \geq 0, \sum_{i=1}^n x_i \leq 1\}$. Then express that the projection of a point y on the set S_2 as an optimization problem with constraints, and write the KKT conditions for the problem. Bonus: simplify the KKT solutions as much as possible to derive a closed form expression of the projection.

Q 2.17: Endsem Spring 2022-23

Consider the following optimization problem:

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & a^\top x \leq b, \qquad \forall a \in \{a \in \mathbb{R}^n | Ca \leq d\}, \end{aligned}$$

where $C \in \mathbb{R}^{m \times n}$ and $d \in \mathbb{R}^m$. Reformulate the above problem as a convex optimization problem with finitely many constraints. Hint: You may need to use linear programming duality in a slightly non trivial manner.

Q 2.18: Endsem Spring 2022-23

Show that the optimal value of the following optimization problem:

$$\begin{aligned} \text{maximize}_{y \in \mathbb{R}^n} \quad & y^\top x \\ \text{subject to} \quad & ||y||_1 \leq 1 \end{aligned}$$

is equal to $||x||_\infty$. Recall that for $x \in \mathbb{R}^n$, $||x||_1 := \sum_{i=1}^n |x_i|$ and $||x||_\infty := \max_{i \in \{1,2,\ldots,n\}} |x_i|$. Hint: Find the dual of the above optimization problem and compute the optimal dual solution. Explain why strong duality holds for this problem.

Q 2.19: Projection via duality

Let $S_1 = \{x \in \mathbb{R}^n | Gx = h\}$ where $G \in \mathbb{R}^{p \times n}$, $h \in \mathbb{R}^p$ and rank of G is p. Then show that the projection of a point y on the set S is given by

$$\Pi_{S_1}(y) = y - G^\top [GG^\top]^{-1}(Gy - b).$$

Q 2.20: Projection via duality

Let $S_2 = \{x \in \mathbb{R}^n | x \geq 0, \sum_{i=1}^n x_i \leq 1\}$. Then show that the projection of a point y on the set S_2 is given by

$$\Pi_{S_2}(y) = \begin{cases} y, & \text{if} \quad y \geq 0, \sum_{i=1}^n y_i \leq 1, \\ \max(y - \mu^*, 0), & \text{otherwise}, \end{cases}$$

where μ^* is the unique solution of the equation $\sum_{i=1}^n \max(y_i - \mu^*, 0) = 1$. Write the projection as an optimization problem with constraints, and use the KKT conditions to derive the above closed-form expression of the projection.