EE61012 – Spring Semester 2024-25

Homework 1: Convex Optimization in Control and Signal Processing

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Q 1.1: Bounded Set

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence that converges to $x^* \in X$. Show that the set $\{x_n\}_{n\in\mathbb{N}} = \{x_1, x_2, \ldots\} \subseteq X$ is bounded. Recall that a sequence converges if for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $||x^* - x_m|| \le \epsilon$ for all $m \ge n$.

Q 1.2: Open and Closed Set

Is the set $X = \{x \in \mathbb{R}^2 | x_1 + x_2 = 1\}$ an open set? Is it a closed set?

Q 1.3: Intersection of Closed Sets

Prove that intersection of arbitrarily many closed sets is a closed set.

Q 1.4: Convex Hull

Let $C = \{x \in \mathbb{R} | 1 \le x \le 2, 3 \le x \le 4\}$. Find the affine hull, conic hull, and convex hull of C.

Q 1.5: Convex Set

Let $x_0, v \in \mathbb{R}^n$. Let $C = \{x \in \mathbb{R}^n | x = x_0 + \alpha v, \alpha \ge 0\}$ be the set of points that lie on the ray originating from x_0 along the direction v. Is C an affine set? Is it a convex set? Is it a cone?

Q 1.6: Convex Set

Show that the set $C = \{x \in \mathbb{R}^2 | x_1 x_2 \ge 1, x_1 \ge 0, x_2 \ge 0\}$ is a convex set. Hint: If $a, b \ge 0$ and $0 \le \theta \le 1$, then $a^\theta b^{(1-\theta)} \le \theta a + (1-\theta) b$.

Q 1.7: Convex Set

Let $x \in \mathbb{R}^n$ and $F_0, F_1, \ldots, F_n \in \mathbb{S}^n$ where $\mathbb{S}^n$ is the set of symmetric matrices of dimension n. A linear matrix inequality (LMI) $F : \mathbb{R}^n \to \mathbb{S}^n$ is an expression of the form:

$$F(x) := F_0 + \sum_{i\in[n]} x_i F_i \preceq 0_{n\times n}.$$

Show that the set $\{x \in \mathbb{R}^n \mid F(x) \preceq 0_{n\times n}\}$ is a convex set.

Q 1.8: Dual Cones

Let $K \subseteq \mathbb{R}^n$ be a cone. Consider the following set:

$$K^* = \{y \in \mathbb{R}^n | x^\top y \geq 0, \forall x \in K\}.$$

K^* is called the dual cone of the set K. Show that:

a. K^* is a convex set and is a cone.

b. Let $K_1 \subseteq K_2$. Then the respective dual cones satisfy: $K_2^* \subseteq K_1^*$.

Q 1.9: Convex Sets and Cones

Is the set $C = \{(x, y) | \quad ||x||_2^2 \leq y\} \subseteq \mathbb{R}^2$ convex? Is this a cone?

Q 1.10: Union of Convex Sets

Give an example of two convex sets X_1 and X_2 whose union is not convex.

Q 1.11: Convex Set

Let $C \in \mathbb{R}^n$ be a compact (closed and bounded) convex set. For a scalar α, let $C(\alpha)$ be the set of all points in C whose first coordinate is closer to α than any other point. In other words, $C(\alpha) = \{x \in C | |x_1 - \alpha| \leq |y_1 - \alpha|, \quad \forall y \in C\}$. Is $C(\alpha)$ a convex set for any choice of C and α?

Q 1.12: Cone

Let $C = \{x \in \mathbb{R}^2 | x_2 \geq 0, x_2 \geq |x_1|\}$. Show that C is a cone.

Q 1.13: Midsem Spring 2023-24

Determine if the following sets in $\mathbb{R}^2$ are convex sets together with suitable explanation.

- $S_1 = \{(x_1, x_2) | x_2 \geq \exp^{(x_1 - 1)}\}$.
- $S_2 = \{(x_1, x_2) | x_2 \leq x_1(x_1 - 2)(x_1 - 3)\}$.
- $S_3 = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 4, x_1 \geq 0\}$.
- $S_4 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2\}$.
- $S_5 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2, (x_1 - 1)^2 + (x_2 + 1)^2 \leq 5\}$.
- $S_6 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2\} \cup \{x \in \mathbb{R}^2 | (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1\}$.

Q 1.14: Endsem Spring 2023-24

Determine if the following sets are convex sets.

- $S_1 = \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i^2 = 1\}$.
- $S_2 = \{x \in \mathbb{R}^3 | \frac{x_1^2 + 4x_2^2 + 4x_1 x_2}{2x_1 + x_2 + x_3} \leq 10, 2x_1 + x_2 + x_3 > 0\}$.

Q 1.15: Separating Hyperplane

Let $C_1 = \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}$, and let $C_2 = \{x \in \mathbb{R}^2 | x_1 \leq 0, x_2 \leq 0\}$. Find a hyperplane that separates C_1 and C_2.

Q 1.16: Convex Function

Let $x_k \in \mathbb{R}^n, k = 1, 2, \ldots, T$. Let $x = [x_1^\top, x_2^\top, \ldots, x_k^\top]^\top \in \mathbb{R}^{nT}$ where $f : \mathbb{R}^{nT} \to \mathbb{R}$. Let $f(x) = \sum_{k=1}^{T} x_k^\top Q_k x_k$, where each Q_k is positive semi-definite. Then show that f is a convex function.

Q 1.17: Midsem Spring 2023-24

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Show that

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq 0, \quad \forall x, y \in \mathtt{dom} f.$$

Q 1.18: Convex Function

Suppose that $g(x)$ is convex and $h(x)$ is concave. Suppose the domain of both functions is a closed, convex set C such that both $g(x)$ and $h(x)$ are always positive. Prove that the function $f(x) = \frac{g(x)}{h(x)}$ is quasi-convex, i.e., all sub-level sets of f are convex sets.

Q 1.19: Convex Function

Determine whether the following functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ are convex, concave or neither.

a. $f_1(x_1, x_2) = x_1 x_2$ with domain $x_1 > 0, x_2 > 0$.

b. $f_2(x_1, x_2) = -\frac{1}{x_1 x_2}$ with domain $x_1 > 0, x_2 > 0$.

c. $f_3(x_1, x_2) = \frac{x_1^2}{x_2}$ with domain $x_1 \in \mathbb{R}, x_2 > 0$.

d. $f_4(x_1, x_2) = x_1 \log\left(1 + \frac{\beta x_2}{x_1}\right)$ with domain $x_1, x_2 > 0$, and $\beta \in \mathbb{R}$ is a constant.

e. $f_5(x) = -(\sum_{i=1}^{n} x_i^a)^{1/a}$ for $a \in (0, 1)$.

f. $f_6(x) = \min(0.5, x, x^2)$.

g. $f_7(x) = \max_{i \in \{1,2,\ldots,n\}} x_i - \min_{i \in \{1,2,\ldots,n\}} x_i$.

Q 1.20: Convex Function

If f is a convex function, then $g(x, t) = t f(x/t)$ is also a convex function. You may use the epigraph definition of convex functions to show the above.

Q 1.21: Convex Optimization

Formulate an optimization problem to find the largest circle contained in the triangle $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, 3x_1 + 4x_2 \leq 12\}$. Clearly state the decision variables, constraints and cost function. Determine if the problem is a convex optimization problem.

Q 1.22: Convex Optimization

Formulate an optimization problem to find the minimum distance between a circle $X_1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ and the line $X_2 := \{x \in \mathbb{R}^2 \mid x_1 - x_2 = 2\}$. Clearly state the decision variables, constraints and cost function. Determine if the problem is a convex optimization problem.

Q 1.23: Midsem Spring 2022-23

Answer the following questions.

1. Let $f(x) = \frac{x^2+2}{x+2}$ with $\texttt{dom}(f) = (-\infty, -2)$. Is this function convex, concave or neither?
2. Show that a function is convex if and only if its epigraph is a convex set.
3. Consider an optimization problem with cost function $f(x) = -x_1 + x_2^2$ and constraint set $X = \{x \in \mathbb{R}^2 | -x_1^2 - x_2^2 + 4 \leq 0, x_1 + x_2 \geq -2\}$. Explain with justificiation whether this problem is a convex optimization problem or not.

Q 1.24: Midsem Spring 2022-23

Consider the set $X := \{x \in \mathbb{R}^n | ||x - z_0||_2 \leq ||x - z_i||_2, i = 1, 2, \ldots, k\}$ where $z_0, z_1, \ldots, z_k \in \mathbb{R}^n$. Show that this set is a polyhendron and can be written as $X := \{x \in \mathbb{R}^n | Ax \leq b\}$. Find A and b.

Q 1.25: Endsem Spring 2022-23

Consider the following optimization problem:

$$\begin{aligned} \text{maximize}_{x \in \mathbb{R}^1} \quad & 2x - x^2 \\ \text{subject to} \quad & 0 \leq x \leq 3. \end{aligned}$$

Is the above problem a convex optimization problem? Find a globally optimal solution of the above.

Q 1.26: Endsem Spring 2022-23

Determine if the following statements are true or false. If true, justify. If false, give a counter example.

1. A convex optimization problem can have at most one globally optimal solution.
2. A convex optimization problem must have at least one globally optimal solution.
3. Any optimization problem with an unbounded feasible region does not have an optimal solution.

Q 1.27: Endsem Spring 2022-23

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = (x_1 + x_2^2)^2$ where $x = [x_1 x_2]$.

1. Compute the gradient of f.
2. At $x_0 = [0, 1]$, is the direction $d = [1, -1]$ a descent direction, i.e., with directional derivative negative?
3. Find $\alpha > 0$ that minimizes $f(x_0 + \alpha d)$.

Q 1.28: Endsem Spring 2023-24

Consider the optimization problem $\max\{g(y) | f_1(y) \leq 0, f_2(y) \leq 0\}$ with g being a concave function and f_1, f_2 being convex functions. Let $Y_{\texttt{opt}}$ be the set of optimal solutions of the above problem.

Now, suppose the problem $\max\{g(y) | f_1(y) \leq 0\}$ has a unique optimal solution $y^\star$.

If $f_2(y^\star) \leq 0$, then show that $Y_{\texttt{opt}} = \{y^\star\}$.

Q 1.29: Endsem Spring 2023-24

Determine if the following functions are convex functions.

- $f_1(x) = \frac{1}{k} \sum_{i=1}^{k} \log(1 + e^{-a_i^\top x})$ where $x \in \mathbb{R}^n$.
- $f_2(x) = \log(e^{x_1} + e^{x_2})$ for $x \in \mathbb{R}^2$.