EE61012 – Spring Semester 2024-25

Class Test 1: Convex Optimization in Control and Signal Processing

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Problem 1. Determine with justification if the following functions are convex, concave or neither. [$2 \times 2 = 4$ points]

- $f_1(x) = \min(0.5, x, -x^2)$
- $f_2(x) = \sum_{i=1}^{n}(|a_i^\top x| - 1)^2$

Problem 2. Consider the optimal values of the following optimization problems.

$$p_1^* = \min_{x \in X_1} f(x), \qquad p_2^* = \min_{x \in X_2} f(x)$$
$$p_{13}^* = \min_{x \in X_1 \cap X_3} f(x), \qquad p_{23}^* = \min_{x \in X_2 \cap X_3} f(x)$$

Suppose $X_1 \subseteq X_2$ and $p_1^* = p_2^*$. Then, show that $p_{13}^* = p_1^* \implies p_{23}^* = p_2^*$. [3 points]

Problem 3. The table indicates the cost and the nutrients associated with one serving of three types of food. Formulate an optimization problem to determine an optimal diet plan such that the cost is minimized, number of calories is between 2000 and 2250, amount of vitamin received is at least 5000 units, and the amount of sugar is at most 1000 units. The amount of servings for each item is at most 10. [3 points]

Food	Cost	Vitamin	Sugar	Calories
Corn	0.15	107	45	70
Milk	0.25	500	40	121
Bread	0.05	0	60	65

Problem 4 : Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top H x + g^\top x \quad \text{s.t.} \quad Ax = b.$$

[5 points]

Derive a system of linear equations whose solution corresponds to the optimal solution.

Model Solutions

(1) a) $f_1(x) = \min(0.5, x, -x^2)$ is concave.

Note that 0.5 and x are affine functions of x, hence concave.

$-x^2$ is a concave function of x.

minimum of concave functions is concave.

b) $f_2(x) = \sum_{i=1}^{n} (|a_i^T x| - b)^2 = \sum_{i=1}^{n} \left[(a_i^T x)^2 - 2b|a_i^T x| + b^2 \right]$

$|a_i^T x| = \max(a_i^T x, -a_i^T x)$ is convex.

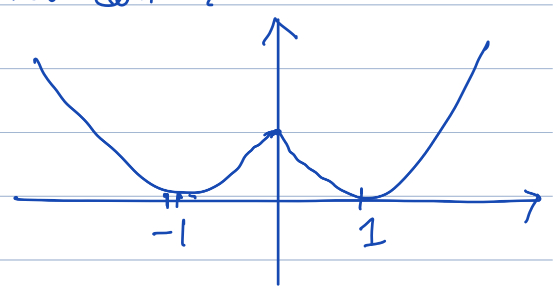
$(a_i^T x)^2$ is convex.

but $(a_i^T x)^2 - |a_i^T x|$ is not guaranteed to be either convex or concave.

<u>Example</u>: let $n=1$, $a=1$, $b=1$

$f(x) = (|x| - 1)^2$

– neither concave nor convex.



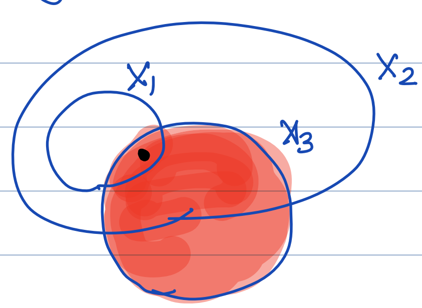
– composition rule does not apply since $f(x) = x^2$ is not monotonically increasing.

(2) Since $X_1 \cap X_3 \subseteq X_1$,

we always have $p_1^* \leq p_{13}^*$

Similarly $X_2 \cap X_3 \subseteq X_2$

Hence $p_2^* \leq p_{23}^*$



Since $X_1 \subseteq X_2$

$X_1 \cap X_3 \subseteq X_2 \cap X_3$

$\Rightarrow p_{13}^* \geq p_{23}^*$.

It is given that $p_1^* = p_2^*$.

Hence, $$p_1^* = p_2^* \leq p_{23}^* \leq p_{13}^*$$

Thus if $p_1^* = p_{13}^*$, then we must have equality, i.e,
$$p_1^* = p_2^* = p_{23}^* = p_{13}^* .$$

Q3 Let x_1 : unit of corn,
x_2 : unit of milk, , $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
x_3 : unit of bread

$$\min_{x \in \mathbb{R}^3} \quad 0.15x_1 + 0.25x_2 + 0.05x_3$$
$$\text{s.t.} \quad 0 \leq x_1 \leq 10, \quad 0 \leq x_2 \leq 10, \quad 0 \leq x_3 \leq 10$$
$$107x_1 + 500x_2 \geq 5000$$
$$45x_1 + 40x_2 + 60x_3 \leq 1000$$
$$2000 \leq 70x_1 + 121x_2 + 65x_3 \leq 2250$$

Q4 $$\mathcal{L}(x,\mu) = \frac{1}{2}x^T Hx + g^T x + \mu^T (Ax - b)$$
$$\nabla_x \mathcal{L}(x,\mu) = \frac{1}{2}(H + H^T)x + g + A^T\mu = 0$$
$$Ax = b.$$

System of linear eqn:
$$\begin{bmatrix} \frac{1}{2}(H+H^T) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} -g \\ b \end{bmatrix}$$

whose solution corresponds to the optimal solution.

– will be discussed in class in more detail.

CLASS TEST-2, 13th Feb 2025

(1) Consider the points $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$ in $\mathbb{R}^2$.

a) Express the convex hull of the above points by a set of linear inequalities.

b) Express the conic hull of the above points by a set of linear inequalities.

[3+3=6 points]

(2) Determine the conditions under which the following optimization problem is a convex optimization problem.

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^T H_0 x + c_0^T x \\ \text{s.t.} \quad & x^T H_1 x + c_1^T x \leq 0 \\ & x^T H_2 x + c_2^T x + d_2 = 0 \end{aligned}$$

[4 points]

(3) Consider an optimization problem $\min_{x\in X} c^T x$, $c \neq 0$, $x \in \mathbb{R}^n$. Let X be closed and convex.

If x^* is an optimal solution of the above problem, then show that x^* cannot be an interior point of the set X.

[5 points]

(4) Consider the LP

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \ : \mu_1 \\ & 2x_1 + 2x_2 = 3 \ : \mu_2 \end{aligned}$$

[5 points]

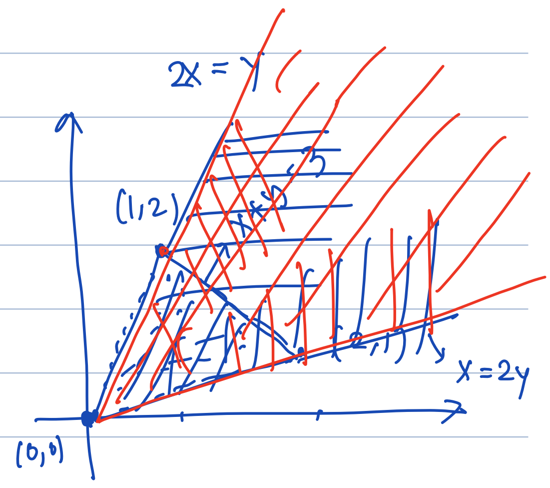
Derive its dual. Determine whether the primal & dual are infeasible or unbounded.

SOLUTION

① a) convex hull

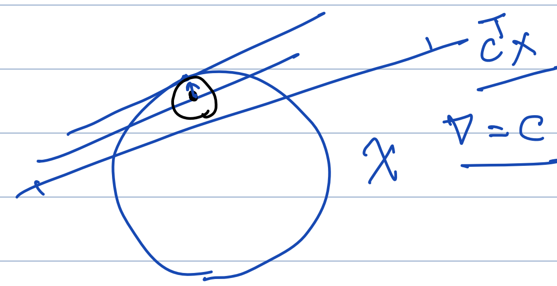
$$\{x \in \mathbb{R}^2 \mid 2x \geq y,\ x \leq 2y,\ x+y \leq 3\}$$

b) conic hull $\{x \in \mathbb{R}^2 \mid 2x \geq y,\ x \leq 2y\}$



② H_0, H_1 should be positive semidefinite.
H_2 should be zero.

③ Suppose x^* is an interior point.
Then $\exists\ r > 0$ s.t.
$B(x^*, r) \subseteq X$.



$\bar{x} = x^* - \delta c$, $\delta > 0$, then $\underline{c^T \bar{x}} = c^T x^* - \delta \underbrace{c^T c}_{+ve} < c^T x^*$

We need to shoose δ such that $\bar{x} \in X$.
If δ is such that $\|\bar{x} - x^*\|_2 = \|\delta c\|_2 \leq r$, then
$\bar{x} \in B(x^*, r)$, and therefore $\bar{x} \in X$.
We choose $\delta = \frac{1}{2}\frac{r}{\|c\|_2}$, $\|\delta c\|_2 = \frac{r}{2} < r$.
Therefore x^* can not be an optimal solution.
Hence no interior point can be an optimal solution.

(4). primal is infeasible.

$$\mathcal{L}(x,\mu) = x_1 + 2x_2 + \mu_1(x_1+x_2-1) + \mu_2(2x_1+2x_2-3)$$

$$= x_1\underline{[1+\mu_1+2\mu_2]} + x_2\underbrace{[2+\mu_1+2\mu_2]} - \mu_1 - 3\mu_2.$$

$$\inf_x \mathcal{L}(x,\mu) = \begin{cases} -\mu_1 - 3\mu_2 & \text{if } 1+\mu_1+2\mu_2=0 \ \& \ 2+\mu_1+2\mu_2=0 \\ -\infty & \text{, otherwise} \end{cases}$$

dual:

$$\max_\mu d(\mu) = \max_\mu -\mu_1 - 3\mu_2$$

$$\text{s.t. } 1+\mu_1+2\mu_2=0$$

$$2+\mu_1+2\mu_2=0$$

the dual is also infeasible.